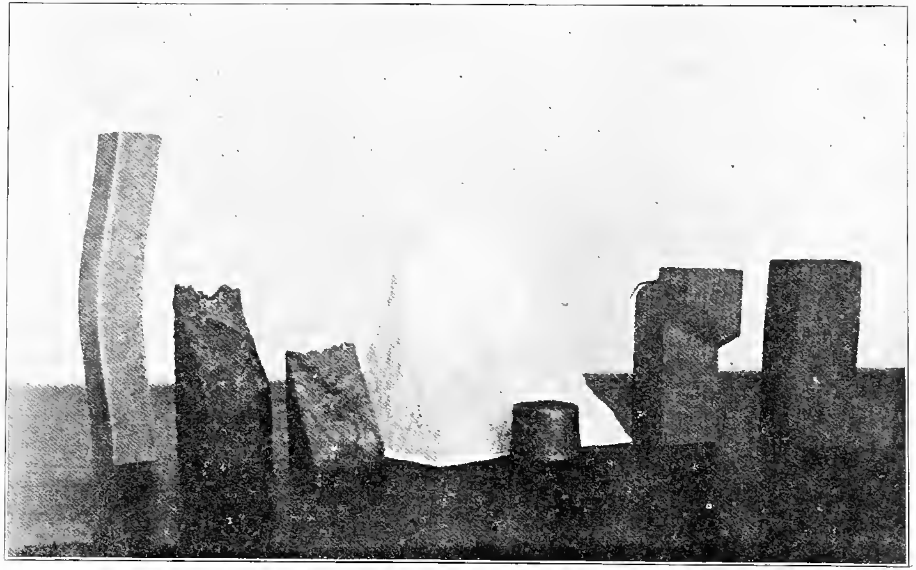

PLATE I.

STEEL CAST IRON SANDSTONE WOOD CAST IRON

THE ACTION OF MATERIALS UNDER STRESS

OR

STRUCTURAL MECHANICS

COMPRISING THE

STRENGTH AND RESISTANCE OF MATERIALS AND ELEMENTS OF STRUCTURAL DESIGN

WITH EXAMPLES AND PROBLEMS

BY

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ANN ARBOR, MICH.:

PRINTED FOR THE AUTHOR

1897.

BY THE SAME AUTHOR.

Graphics for Engineers, Architects and Builders. A manual for designers, and a text-book for scientific schools.

Trusses and Arches; Analyzed and Discussed by Graphical Methods. In three parts—published separately.

Part I. Roof Trusses: Diagrams for Steady Load, Snow and Wind. 8vo., 80 pp., 3 folding plates. Revised Edition. $1.25.

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THE INLAND PRESS, ANN ARBOR.

PREFACE.

The author, in teaching for many years the subjects embraced in the following pages, has found it advantageous to take at first but a portion of what is included in the several chapters, and, after a general survey of the field, to return and extend the investigation more in detail. Some of the sections, therefore, are printed in smaller type and can be omitted at first reading. A few of the special investigations may become of interest only when the problems to which they relate occur in actual practice.

It is hoped that this book will be serviceable after the class-room work is concluded, and reference is facilitated by a more compact arrangement of the several matters than the course suggested above would give. The attempt has been made to deal with practicable cases, and the examples for the most part are shaped with that end in view. A full index will enable one to find any desired topic.

The treatment of the subject of internal stress is largely graphical. All the constructions are simple, and the results, besides being useful in themselves, shed much light on various problems. The time devoted to a careful study of the chapter in question will be well expended.

The notation is practically uniform throughout the book, and is that used by several standard authors. Forces and moments are expressed by capital letters, and unit loads and stresses by small letters. The co-ordinate x is measured along the length of a piece, the co-ordinate y in the direction of variation of stress in a section, and z is the line of no variation of stress, that is, the line parallel to the moment axis.

One who has mastered the subjects discussed here can use the current formulas, the pocket-book rules, and tables, not blindly, but with discrimination, and ought to be prepared to design intelligently.

Mr. Albert E. Greene has rendered much assistance in preparing the material for publication.

TABLE OF CONTENTS.

INTRODUCTION.

1. **External Forces.**—The engineer, in designing a new structure, or critically examining one already built, determines from the conditions of the case the actual or probable external forces which the structure is called upon to resist. He may then prepare, either by mathematical calculations or by graphical methods,* a sheet which shows the maximum and minimum direct forces of tension and compression which the several pieces or parts of the structure are liable to experience, as well as the bending moments on such parts as are subjected to them.

These forces and moments are determined from the requirements of equilibrium, if the pieces are at rest. For forces acting in one plane, a condition which suffices for the analysis of most cases, it is necessary that, for the structure as a whole, as well as for each piece, there shall be no tendency to move up or down, to move to the right or left, or to rotate. These limitations are usually expressed in Mechanics as, that the sum of the X forces, the sum of the Y forces, and the sum of the moments shall each equal zero.

If the structure is a machine, the forces and moments in action at any time, and their respective magnitudes, call for a consideration of the question of acceleration or retardation of the several parts and the additional maximum forces and moments called into action by the greatest rate of change of motion at any instant. Hence the weight or mass of the moving part or parts is necessarily taken into account.

Finally, noting the rapidity and frequency of the change of force and moment at any section of any piece or connection, the engineer selects, as judgment dictates, the allowable stresses of the several kinds per square inch, making allowance for the effect of impact, shock and vibration in intensify-

*See the author's "Trusses and Arches," considered Graphically: Part I., Roof Trusses; Part II., Bridge Trusses; Part III., Arches. New York, Wiley & Sons.

ing their action, and proceeds to find the necessary cross-sections of the parts and the proportions of the connections between them. As all structures are intended to endure the forces and vicissitudes to which they are usually exposed, the allowable unit-stresses, expressed in pounds per square inch, must be *safe stresses.*

It is largely with the development of the latter part of this subject, after the *forces* have been found to which the several parts are liable, that this book is concerned.

2. **Ties, Struts and Beams.**—There are, in general, three kinds of pieces in a frame or structure; ties or tension members; columns, posts and struts or compression members; and beams, which support a transverse load and are subject to bending and its accompanying shear. A given piece may also be, at the same time, a tie and a beam, or a strut and a beam, and at different times a tie and a strut.

3. **Relation of External Forces to Internal Stresses.**—The forces and moments which a member is called upon to resist, and which may properly be considered as *external* to that member, give rise to actions between all the *particles* of material of which such a member is composed, tending to move adjacent particles from, towards or by one another, and causing change of form. There result *internal stresses*, or resistances to displacement, between the several particles.

These internal stresses, or briefly stresses, must be of such kind, magnitude, distribution and direction, at any imaginary *section* of a piece or structure, that their resultant force and moment will satisfy the requirements of equilibrium or change of motion with the external resultant force and moment at that section; and no stress per square inch can, for a correct design, be greater than the material will safely bear. Hence may be determined the necessary area and form of the cross-section at the critical points, when the resultant forces and moments are known.

4. **Internal Stress.**—There are three kinds of stress, or action of adjacent particles one on the other, to which the particles of a body may be subjected, when external forces and its own weight are considered, viz.: *tensile* stress, tending to remove one particle farther from its neighbor, and mani-

fested by an accompanying stretch or elongation of the body; *compressive* stress, tending to make a particle approach its neighbor, and manifested by an accompanying shortening or compression of the body; and *shearing* stress, tending to make a particle move or slide laterally with reference to an adjacent particle, and manifested by an accompanying distortion. Whether the stress produces change of form, or the attempted change of form gives rise to internal stresses as resistances, is of little consequence; the stress between two particles and the change of position of the particles are always associated, and one being given the other must exist.

5. **Tension and Shear, or Compression and Shear.**—If the direction of the stress is oblique, that is, not normal or perpendicular, on any section of a body, the stress may be resolved into a tensile or compressive stress normal to that section, and a tangential stress along the section, which, from its tendency to cause sliding of one portion of the body by or along the section, has been given the name of shear, from the resemblance to the action of a pair of shears, one blade passing by the other along the opposite sides of the plane of section. Draw two oblique and directly opposed arrows, one on either side of a straight line representing the trace of a sectional plane, decompose those oblique stresses normally and tangentially to the plane, and notice the resulting directly opposed tension or compression, and the shear. Hence tension and shear, or compression and shear, may be found on any given plane in a body, but tension and compression cannot simultaneously occur at one point in a given area.

6. **Sign of Stress.**—Ties are usually slender members; struts have larger lateral dimensions. Longitudinal tension tends to diminish the cross-section of the piece which carries it, and hence may conveniently be represented by —, the negative sign; longitudinal compression tends to increase the cross-sectional area and may be called + or positive. Shear, being at right angles to the tension and compression in the preceding illustration, has no sign; and lies, in significance, between tension and compression. If a rectangular plate is pulled in the direction of two of its opposite sides and com-

pressed in the direction of its other two sides, there will be some shearing stress on every plane of section except those parallel to the sides, and nothing but shear on two certain oblique planes, as will be seen later.

7. **Unit Stresses.**—These internal stresses are measured by units of pounds and inches by English and American engineers, and are stated as so many pounds of tension, compression or shear per square inch, called unit tension, compression or shear. Thus, in a bar of four square inches cross-section, under a total pull of 36,000 pounds centrally applied, the internal unit tension is 9000 pounds per square inch, provided the pull is uniformly distributed on the particles adjacent to any cross-section. If the pull is not central or the stress not uniformly distributed, the average or mean unit tensile stress is still 9000 pounds.

If an oblique section of the same bar is made, the total *force* acting on the particles adjacent to the section is the same as before, but the area of section is increased; hence the unit stress, found by dividing the force by the new area, is diminished. The stress will also be oblique to the section, as its direction must be that of the force. When the unit stress is not normal to the plane of section on which it acts, it can be decomposed into a normal unit tension and a unit shear. See § 180.

When the stress varies in magnitude from point to point, its amount on any very small area (the infinitesimal area of the Calculus) may be divided by that area, and the quotient will be the unit stress, or the amount which would exist on a square inch, if a square inch had the same stress all over it as the very small area has.

8. **Unit Stresses on Different Planes not to be Treated as Forces.**—It will be seen, upon inspection of the results of analyses which come later, that unit stresses acting on different planes must not be compounded and resolved as if they were forces. But the entire stress upon a certain area, found by multiplying the unit stress by that area, is a force, and this force may be compounded with other forces or resolved, and the new force may then be divided by the new area of action, and a new unit stress be thus found.

Some persons may be assisted in understanding the analysis of problems by representing in a sketch, or mentally, the unit stresses at different parts of a cross-section by ordinates which make up, in their assemblage, a volume. This volume, whose base is the cross-section, will represent or be proportional to the total force on the section. The position of the resultant force or forces, *i. e.*, traversing the centre of gravity of the volume, the direction and law of distribution of the stress are then quite apparent.

CHAPTER I.

ACTION OF A PIECE UNDER DIRECT FORCE.

9. **Change of Length under an Applied Force.**—Let a uniform bar of steel have a moderate amount of tension applied to its two ends. It will be found, upon measurement, to have increased in length uniformly throughout the measured distance. Upon release of the tension the stretch disappears, the bar resuming its original length. A second application of the same amount of tension will cause the same elongation, and its removal will be followed by the same contraction to the original length. The bar acts like a spring. This elastic elongation (or shortening under compression) is manifested by all substances which have definite form and are used in construction; and it is the cause of such changes of shape as structures, commonly considered rigid, experience under changing loads. The product of the elongation (or shortening) into the mean force that produced it is a measure of the work done in causing the change of length. As the energy of a moving body can be overcome only by work done, the above product becomes of practical interest in structures where moving loads, shocks and vibrations play an important part.

10. **Modulus of Elasticity.**—If the bar of steel is stretched with a greater force, but still a moderate one, it is found by careful measurement that the elongation has increased with the force; and the relationship may be laid down that the *elongation* per linear inch *is directly proportional to the unit stress* on the cross-section per square inch.

The ratio of the unit stress to the elongation per unit of length is denoted by E, which is termed the *modulus of elasticity* of the material, and is based, in English and American books, upon the pound and inch as units. If P is the total tension in pounds applied to the cross-section S measured in

square inches, λl is the elongation in inches, produced by the tension, in the previously measured length of l inches,

$$E = \frac{P}{S\lambda}; \quad \lambda = \frac{P}{S\,E} \text{ per inch.}$$

Hence, if E has been determined for a given material, the stretch of a given bar under a given unit stress is easily found.

Example.—A bar of 6 sq. in. section stretches 0.09 in., in a measured length of 120 in., under a pull of 120,000 lbs.

$$E = \frac{120000 \times 120}{6 \times 0.09} = 26,700,000.$$

If the stress were compressive, a similar modulus would result, which will be shown presently to agree with the one just derived.

If one particle is displaced laterally with regard to its neighbor, under the action of a shearing stress, a modulus of shearing elasticity will be obtained, denoted by C, the ratio of the displacement or distortion to the unit shear which accompanies it.

11. **Stress-Stretch Diagram.**—The elongations caused in a certain bar, or the stretch per unit of length, may be plotted as abscissas, and the corresponding forces producing the stretch, or the unit stresses per square inch, may be used as ordinates, defining a certain curve, as represented in Fig. 1. This curve can be drawn on paper by the specimen itself, when in the testing machine, if the paper is moved in one direction to correspond with the movement of the poise on the weighing arm, and the pencil is moved at right angles by the stretch of the specimen.

A similar diagram can be made for a compression specimen, and may be drawn in the diagonally opposite quadrant. Pull will then be rightly represented as of opposite sign to thrust, and extension will be laid off in the opposite direction to shortening or compression.

12. **Work of Elongation.**—If the different unit stresses applied to the bar are laid off on O Y as ordinates and the resulting stretches per unit of length on O X as abscissas, the portion of the diagram near the origin will be found to be a

straight line, more or less oblique, according to the scale by which the elongations are platted. The elongation varies directly as the unit stress, beginning with zero. Hence the mean force is $\frac{1}{2}$ P, and the work done in stretching a given bar with a given force, if the limit of elastic stretch is not exceeded, is

$$\text{Work} = \frac{P}{2} \cdot \lambda l = \frac{P^2 l}{2 E S}.$$

It may be seen that the work done in stretching the bar is represented by the area included between the base line or axis, the curve O A and the ordinate at A. It also appears that E may be looked upon as the tangent of the angle X O A. A material of greater resistance to elongation will give an angle greater than X O A and *vice versa.*

Example.—A bar 20 ft. = 240 in. long and 3 sq. in. in section is to have a stress applied of 10,000 lbs. per sq. in.; if E = 28,000,000, the work done on the bar will be

$$\frac{10{,}000 \cdot 30{,}000 \cdot 240}{2 \cdot 28{,}000{,}000} = 1{,}286 \text{ in. lbs.},$$

and the stretch will be $1{,}286 \div 5{,}000 = 0.257$ in.

13. **Permanent Set.**—While the unit stress may be gradually increased with corresponding increase of stretch, and apparently complete recovery of original length when the bar is released, there comes a time when very minute and delicate measurements show that the elongation has increased in a slightly greater degree than has the stress. The line O A at and beyond such a point must therefore be a curve, concave to the axis of X. If the piece is now relieved from stress, it will be found that the bar has become permanently lengthened. The amount of this increase of length after removal of stress is called set, or *permanent set*, and the unit stress for which a permanent set can first be detected is known as the *elastic limit*. As the elongation itself is an exceedingly small quantity, even when measured in a length of many inches, and the permanent set is, in the beginning, a quantity far smaller and hence more difficult of determination, the place where the straight line O A first begins to curve is naturally hard to locate, and the accurate elastic limit is therefore uncertain.

Some contend that O A itself is a curve of extreme flatness. The common or commercial elastic limit lies much farther up the curve, where the permanent set becomes decidedly notable.

If, after a certain amount of permanent set has occurred in a bar, and the force which caused it has been removed, a somewhat smaller force is repeatedly applied to the bar, the piece will elongate and contract elastically to the new length, *i. e.*, old length plus permanent set, just as if the unit stress were below the elastic limit.

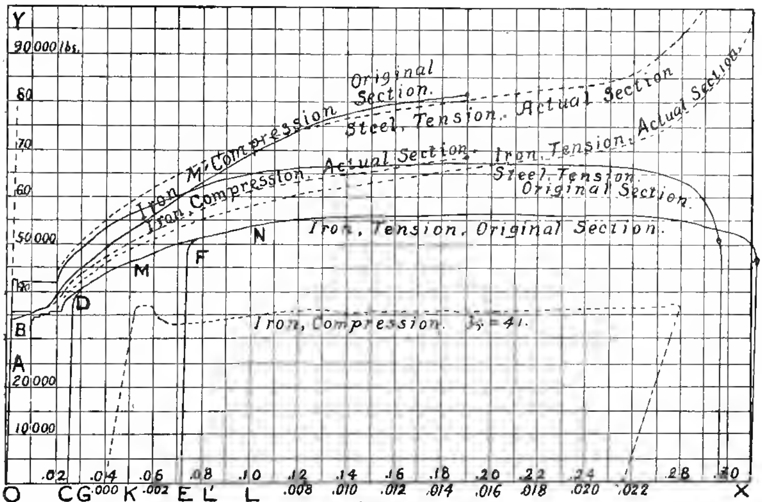


Fig. 1.

14. **Yield Point.**—The unit stress increasing, the elongation increases and the permanent set increases until a unit stress B is reached, known as the *yield point* (or *commercial elastic limit*, or common elastic limit), which causes the bar to yield or draw out without increase of force, and, as the section must decrease, apparently with decreasing power of resistance. There will then be a break of continuity in the graphic curve. A decided permanent elongation of the bar takes place at this time—sufficient to dislodge the scale from the surface of a steel bar, if left as it comes from the rolls or hammer. The weighing beam of the testing machine falls, from the diminished resistance just referred to, and remains station-

ary while the bar is elongating for a sensible interval of time. Hence, for steel, the yield point, or common elastic limit, is easily determined by what is known as the "drop of the beam." The remainder of the curve, up to the breaking point, is shown in the figure.

15. **Elastic Limit Raised.**—For stresses above the yield point also, a second application and release of stress will give an elastic elongation and contraction as before the occurrence of set, as shown by lines C D, E F, so that a new elastic limit may be said to be established. The stretch due to any given stress may be considered to be the elastic elongation plus the permanent set; and, for repetitions of lesser forces, the bar will give a line parallel to O A, as if drawn from a new origin on O X, distant from O the amount of the permanent set.

The effects of a period of rest and of application of compression are spoken of later.

If the line O A is prolonged upwards, it will divide each abscissa into two parts, of which that on the left of O A will be the elastic stretch, and that on the right of O A the permanent set for a given unit stress.

16. **Work of Elongation, for Stress above Yield Point.**—The area below the curve, and limited by any ordinate G D, will be the work done in stretching the bar with a force represented by the product of that ordinate into the bar's cross-section, and if a line be drawn from the upper end of that ordinate parallel to O A, the triangle C D G will give the work done in elastic stretch and the quasi-parallelogram O B D C will show the permanent work of deformation done on the bar. It should be remembered that, as the bar stretches, the section decreases, and that the unit stress cannot therefore be strictly represented by $P \div S$, if S is the original cross-section. The error is not of practical consequence for this discussion.

17. **Ultimate or Breaking Strength.**—If the force applied in tension to the bar is increased, a point will next be reached where a repeated application of the same *force* causes a successive increase in the permanent elongation. As this phenomenon means a gradual drawing out, final failure by pulling asunder is only a matter of a greater or less number

of applications of the force. While the bar is apparently breaking under this force, the rapid diminution of cross-section near the breaking point actually gives a constantly rising unit stress, as will be seen by the dotted curve in the figure.

If, however, the force is increased without pause from the beginning, the breaking force will be higher, as might be expected; since much work of deformation is done upon the bar before fracture. The bar would have broken under a somewhat smaller force, applied statically for a considerable time.

The elongation of the bar was uniform per unit of length during the earlier part of the test. There comes a time when a portion or section of the bar, from some local cause, begins to yield more rapidly than the rest. At once the unit stress at that section becomes greater than in the rest of the bar, by reason of decrease of cross-section, and the drawing out becomes intensified, with the result of a great local elongation and necking of the specimen and an assured final fracture at that place. If the bar were perfectly homogeneous, and the stress uniformly distributed, the bar ought to break at the middle of the length, where the *flow* of the metal is most free.

It is customary to determine, and to require by specification, in addition to elastic limit and ultimate strength (on one continuous application of increasing load), the per cent. of elongation after fracture (which is strictly the permanent set) in a certain original measured length, usually eight inches, and the per cent. of reduction of the original area, after fracture, at the point of fracture. As the measured length must include the much contracted neck, the *average* per cent. of elongation is given under these conditions. A few inches excluding the neck would show less extension and an inch or two at the neck would give a far higher per cent. of elongation. The area between the axis of X, the extreme ordinate and the curve will be the work of fracture, if S is considered constant, and will be a measure of the ability of the material to resist shocks, blows and vibrations before fracture. It is indicative of the toughness or ductility of the material.

The actual curve described by the autographic attachment to a testing machine is represented by the full line; the real relation of stress per square inch to the elongation produced,

when account is taken of the progressive reduction of sectional area, is shown by the dotted line. The yield point, or common elastic limit, is very marked, there appearing to be a decided giving way or rearrangement of the particles at that value of stress. The true elastic limit is much below that point. The other curves represent the behavior of wrought iron as marked.

18. **Effect of a Varying Cross-Section.**—If a test specimen is reduced to a smaller cross-section, by cutting out a curved surface, for only a short distance as compared with its transverse dimensions, it will show a greater unit breaking stress, as the metal does not flow freely, and lateral contraction of area is hindered. But, if the portion of reduced cross-section joins the rest of the bar by a shoulder, the apparent strength is reduced, owing to a concentration of stress on the particles at the corner as the unit stress suddenly changes from the smaller value on the larger section to the greater unit stress on the smaller cross-section.

19. **Compression Curve.**—A piece subjected to compression will shorten, the particles being forced nearer together, and the cross-section will increase. It might be expected, and is found by experiment to be the case, that, in the beginning, the resistance of the particles to approach would be like their resistance to separation under tension, so that the tension diagram might be prolonged through the origin into the third quadrant, reversing the sign of the ordinate which represents unit stress and of the abscissa which shows the corresponding change of length. As this part of the diagram is a straight line, it follows that the value of E, the elastic modulus for compression, is the same as that for tension. After passing the elastic limit the phenomena of compression are not so readily determined, as fracture or failure by compressive stress is not a simple matter, and the increase of sectional area in a short column of ductile material will interfere with the experiment. In long columns and with materials not ductile, failure takes place in other ways, as will be explained later.

The compression curve is here shown in the same quadrant with the tension curve for convenience and comparison.

It will be seen that, when the tension and compression curves are corrected for change of cross-section, they are practically the same; and that the resistance per square inch really increases up to fracture.

20. Resilience.—By definition, § 10, if f is the unit stress per square inch and λl the stretch of a bar of length l, in inches, the modulus of elasticity $E = f \div \lambda$, provided f does not exceed the elastic limit. Also the work done in stretching a bar to that elastic limit by a force P, gradually applied, that is, beginning with zero and increasing with the stretch, is the product of the *mean* force, $\frac{1}{2}$ P, into the stretch, or

$$\text{Work done} = \tfrac{1}{2}\, P\lambda l = \frac{fS}{2} \cdot \frac{fl}{E} = \frac{f^2}{2\,E} \cdot Sl.$$

Sl is the volume of the bar; $f^2 \div 2E$ is called the modulus of resilience, when f is the elastic limit, or sometimes the maximum safe unit stress. This modulus depends upon the quality of the material, and, as it is directly proportional to the amount of work that can safely be done upon the bar by a load, it is a measure of the capacity of a certain material for resisting or absorbing shock and impact without damage. For a particular piece, the volume Sl is also a factor as above. A light structure will suffer more from sudden or rapid loading than will a heavier one of the same material, if proportioned for the same unit stress.

21. Work Done Beyond the Elastic Limit.—The work done in stretching a bar to any extent is, in Fig. 1, the area in the diagram between the curve from the origin up to any point, the ordinate to that point and the axis of abscissas, provided the ordinate represents P, and the abscissa the total stretch.

Further, it may be seen from the figure that, if a load applied to the bar has exceeded the yield point, the bar, in afterwards contracting, follows the line D C or F E; and, upon a second application of the load, the right triangle of which this line is the hypothenuse will be the work done in the second application, a smaller quantity than for the first application. But, if the load, in its second and subsequent applications, possesses a certain amount of energy, by reason of

not being gently or slowly applied, this energy may exceed the area of the triangle last referred to, with the result that the stress on the particles of the bar may become greater than on the first application. Indeed it is conceivable that this load may be applied in such a way that the resulting unit stress may mount higher and higher with repeated applications of load, until the bar is broken with an apparent unit stress $P \div S$, far less than the ultimate strength, and one which at first was not much above the yield point. If the load in its first application is above the yield point of the material, and it is repeated continuously, rupture will finally occur.

What is true for tensile stresses is equally true for compressive stresses, except that the ultimate strength of ductile materials under compression is uncertain and rather indefinite.

22. **Sudden Application of Load.**—If a wrought iron or soft steel rod, 10 feet = 120 inches long, and one square inch in section, with $E = 28,000,000$, is loaded gradually with 12,000 pounds longitudinal tension, its stretch will be $12,000 \times 120 \div 28,000,000 = 0.05$ inches, and the work done in stretching the bar will be $6,000 \times 0.05 = 300$ in. pounds.

But, if the 12,000 lbs. is suddenly applied, as by the extremely rapid loading of a structure of which the rod forms a part, or the quick removal of a support which held this weight at the lower end of the rod, the load will cause a greater stretch at first, after which the rod will contract and then undergo a series of longitudinal vibrations of decreasing amplitudes, finally settling down to a stretch of 0.05 inches when the extra work of acceleration has been absorbed.

To ascertain what suddenly applied force will produce, at the most, a stretch of 0.05 in., and hence the same unit stress as a quiescent load (for stretch and stress are directly related), observe that the work done by the application of a gradually applied load is represented by the area of a triangle of 300 in. pounds, the force increasing from zero to P, with a mean value of $\frac{1}{2}$ P. A constant force during the stretch gives a rectangle for the area representing work done, and can be only $300 \div 0.05 = 6,000$ lbs., or half as much as before. The work of acceleration on the mass of the bar is neglected.

Stresses produced by moving loads on a structure are intermediate in effect between these two extremes, depending upon rapidity or suddenness of loading. Hence it is seen why the practice has arisen of limiting stresses due to moving loads *apparently* to only one-half of the values permitted for those caused by static loading.

For resilience or work done in deflection of beams, see § 114.

23. Granular Substances Under Compression.—Failure by Shearing on Oblique Planes. Blocks of material, such as cast-iron, sandstone or concrete, when subjected to compression, frequently give way by fracturing on one or more oblique planes which cut the block into two wedges, or into pyramids and wedges. The pyramids may overlap, and their bases are in the upper and lower faces of the block. This mode of fracture, peculiar to granular substances, of comparatively low shearing resistance, can be discussed as follows:

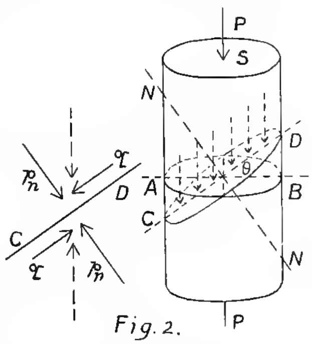


Fig. 2.

If a short column, Fig. 2, of cross-section S is loaded centrally with P, the unit compression on the right section will be $p_1 = P \div S$, and, if the short column gives way under this load, this value of p_1 is commonly considered the crushing strength of the material. While it doubtless is the *available* crushing strength of this specimen, it may by no means represent the maximum resistance to crushing under other conditions.

If $p_1 = P \div S$ is the unit thrust on the right section, it is seen, from § 180, that, on a plane making an angle θ with the right section, the normal unit stress, $p_n = p_1 \cos^2 \theta$, and the tangential unit stress $q = p_1 \sin \theta \cos \theta$. If m = coefficient of frictional resistance of the material to sliding, the resistance per square inch to sliding along this oblique plane will be $mp_n = mp_1 \cos^2 \theta$, and the portion of the unit shearing stress tending to produce fracture along this plane will be $q - mp_n = p_1 (\sin \theta \cos \theta - m \cos^2 \theta)$.

Fracture by shearing, if it occurs, will take place along

that plane for which the above expression is a maximum, or $d(q - mp_n) \div d\theta = 0$. Differentiating relatively to θ,

$$p_1(\cos^2\theta - \sin^2\theta + 2m\sin\theta\cos\theta) = 0.$$
$$\text{Sin}^2\theta - 2m\sin\theta\cos\theta + m^2\cos^2\theta = \cos^2\theta + m^2\cos^2\theta;$$
$$\sin\theta - m\cos\theta = \cos\theta\sqrt{(1 + m^2)}.$$
$$\frac{\sin\theta}{\cos\theta} = \tan\theta = m + \sqrt{(1 + m^2)}.$$

If m were zero, θ max. would be 45°. Therefore the plane of fracture always makes an angle greater than 45° with the right section. As θ may be negative as well as positive, fracture tends to form pyramids or cones.

Example.—A rectangular prism of cast-iron, 2 in. high and square section = 1.05 sq. in., sheared off under a load of 97,000 lbs., or 92,380 lbs. compression per sq. in. of cross-section, at an angle whose tangent was 1.5, or 56° 19′.

$$1.5 = m + \sqrt{(1 + m^2)};\ 2.25 - 3m + m^2 = 1 + m^2;\ m = 0.42.$$
$$\text{Sin}\,\theta = 0.8921, \cos\theta = 0.5546, \sin\theta\cos\theta = 0.495, \cos^2\theta = 0.308.$$
$$92,380(0.495 - 0.42 \times 0.308) = 33,800 \text{ lbs.}$$

The coefficient of friction is 0.42, and the shear 33,800 lbs. per sq. in. The crushing strength of a short block would have exceeded considerably the above 92,400 lbs.

Since this deviation of the plane of fracture from 45° is due to a resistance analogous to friction, it follows that, when a column of granular material, and of moderate length, gives way by shearing, the value p_1 will be only that compressive stress which is compatible with the unit shearing strength, while its real compressive strength in large blocks will be much higher.

The same phenomenon is exhibited by blocks of sandstone and of concrete. Tests of cubes and flat pieces yield higher results than do those of prisms of the same cross-section and having a considerably greater height. See plate I. for views of failure by shearing of cast-iron and sandstone.

24. Ductile Substances Under Compression.—Wrought iron, and soft and medium steel, as well as other ductile substances, tested in short blocks in compression, bulge or swell in transverse dimensions, and do not fracture. Hence the ultimate compressive strength is indefinite.

25. Fibrous Substances Under Compression.—Wood and fibrous substances which have but small lateral cohesion of the fibres, when compressed in short pieces in the direction of the same, separate into component fibres at some irregular section, and the several fibres fail laterally and crush. See Plate I. for crushing of wood.

26. Vitreous Substances Under Compression.—Vitreous substances, like glass and vitrified bricks, tend to split in the direction of the applied force.

27. Resistance of Large Blocks.—The resistance per square inch of a cube to compression will depend upon the size of the cube. As the unit stress and the resulting deformation are associated, as noted in § 4, it follows that the unit compressive stress will be greatest at the centre of the compressed surface and least at the free edges where lateral movement of the particles is less restrained. Hence, the larger the cube, the greater the mean or apparent strength per square inch. Large blocks of stone, therefore, have a greater average sustaining power per square inch than is indicated by small test specimens, other things being equal.

The same inference can be drawn as to resistance of short pieces to tension as compared with longer pieces of the same cross-section.

A uniform compression over any cross-section of a large post or masonry pier, when the load is centrally applied to but a small portion of the top can be realized only approximately; the same thing is probably true of the foundation below the pier. The resisting capacity of the material, if earth, is thereby enhanced; for the tendency to escape laterally at the edges of the foundation is not so great as would be the case if the load were equally severe over the whole base.

Beveling the edges of the compressed face of a block will increase the apparent resistance of the material by taking the load from the part least able to stand the pressure. The unloaded perimeter may then act like a hoop to the remainder.

Examples.—1. A round bar, 1 in. in diameter and 10 ft. long stretches 0.06 in., under a pull of 10,000 lbs. What is the value of E? What is the work done? 25,464,733; 300 in. pounds.

2. If the elastic limit of the bar is reached by a tension of 30,000 lbs. per sq. in., what is the work done or the resilience of the bar? 1,666 in. pounds.

3. An iron rod, $E = 29,000,000$, hangs in a shaft 1,500 ft. deep. What will be the stretch? 1.55 in.

4. A certain rod, 22 ft. long, and having $E = 28,000,000$, is to be adjusted by a nut of 8 threads to the inch to an initial tension of 10,000 lbs. per sq. in. If the connections were rigid, how

much of a revolution ought to be given to the nut after it fairly bears? 0.75.

5. Can a weight of 20,000 lbs. be lifted by cooling a steel bar, 1 in. sq. from 212° to 62° F.? Co-efficient of expansion = .0012 for 180°; E = 29,000,000.

6. A steel eyebar, 80 in. long and 2 in. sq., fits on a pin at each end with 1-50 in. play. What will be the tension in the bar, if the temperature falls 75° F. and the pins do not yield?

7,250 lbs. per sq. in.

7. A cross grained stick of pine, one sq. in. in section sheared off at an angle of about 66° with the right section under a compressive load of 3,200 lbs. If the coefficient of friction is 0.5, what is the unit shearing stress of the section, the actual irregular area being 2.9 sq. in.? 800 lbs.

CHAPTER II.

MATERIALS.

28. **Growth of Trees.**—Trees from which lumber is cut grow by the formation of woody fibre between the trunk and the bark, and each annual addition is more or less distinctly visible as a ring. The sap circulates through the newer wood, and in most trees the heart wood, as it is called, can be easily distinguished from the sap wood. The former is considered more strong and durable, unless the tree has passed its prime. The heart then deteriorates. Sap wood, in timber exposed to the weather, is the first to decay.

Branches increase in size by the addition of rings, as does the trunk; hence a knot is formed at the junction of the branch with the trunk. The knot begins where the original bud started, and increases in diameter towards the exterior of the trunk, as the branch grows. The grain of the annual growth, formed around the junction of the branch with the trunk, is much distorted. Hence timber that contains large knots is very much weaker than straight grained timber. Even small knots determine the point of fracture when timber is experimentally tested for strength. When a branch happens to die, but the stub remains, and annual rings are added to the trunk, a dead or loose knot occurs in the sawed timber; such a knot is considered a defect, as likely to let in moisture and start decay.

As forest trees grow close together, the branches die successively from below from lack of sunlight; such trees develop straight trunks of but little taper, free from any knots, except insignificant ones immediately around the centre, and yield straight grained, clear lumber. A few trees, like hemlock, sometimes have their fibres running in a spiral, and hence yield cross-grained timber. Trees that grow in open spaces have large side limbs, and the lumber cut from them has large knots.

29. **Shrinkage of Timber.**—If a log is stripped of its bark and allowed to dry or season, it will be found that the contraction or shrinkage in the direction of the radius is practically nothing. There are numerous bundles or ribbons of hard tissue running radially through the annual rings which appear to prevent such shrinkage. Radial cracks, running in to a greater or less distance, indicate that the several rings have yielded to the tension set up by the tendency to shrink circumferentially. Sawed timber of any size is likely to exhibit these season cracks. Such cracks are blemishes and may weaken the timber when used for columns or beams. By slow drying, and by boring a hole through the axis to promote drying within, the tendency to form season cracks may be diminished.

A board sawed radially from a log will not shrink in width, and will resist wear in a floor. Such lumber is known as *quarter sawed.* A board taken off near the slab will shrink much and will tend to warp or become concave on that side which faced the exterior of the log. For that reason, and because the annual rings have less adhesion than the individual fibres have, all boards exposed to wear, as in floors, should be laid heart-side down.

30. **Decay of Wood.**—Timber exposed to the weather should be so framed together, if possible, that water will not collect in joints and mortises, and that air may have ready access to all parts, to promote rapid drying after rain. The end of the grain should not be exposed to the direct entrance of water, but should be covered, or so sloped that water can run off, and the ends should be stopped with paint. It is well to paint joints before they are put together.

The decay of timber is due to the presence and action of vegetable growths or fungi, the spores of which find lodgment in the pores of the wood, but require air and moisture, with a suitable temperature, for their germination and spread. Hence if timber is kept perfectly dry it will last indefinitely. If it is entirely immersed in water, it will also endure, as air is excluded. Moisture may be excluded from an exposed surface by the use of paint. Unseasoned timber placed where there is no circulation of air will dry-rot rapidly in the interior

of the stick; but the exterior shell will be preserved, since it dries out or seasons to a little depth very soon.

The worst location for timber is at or near the ground surface; it is then continually damp and rot spreads fast.

31. **Preservation of Wood.**—The artificial treatment of timber to guard against decay may be briefly described as the introduction into the pores of some poison or antiseptic to prevent the germination of the spores; such treatment is efficacious as long as the substance introduced remains in the wood.

The most common fluids forced by pressure into the pores of the wood are cresote and zinc chloride; although corrosive sublimate, copper sulphate and other chemicals have been employed. As much as from 12 to 20 pounds of creosote or dead oil of tar have been forced into a cubic foot of timber, adding materially to its weight. From timber partly immersed in water the creosote may wash out to some extent.

Burnettizing is the name given to treatment with zinc chloride, a comparatively cheap process, applied to railway ties, paving blocks and bridge timbers. To prevent the zinc chloride from dissolving out in wet situations, tannin has been added after the zinc, to form with the vegetable albumen a sort of artificial leather, plugging up the pores. Hence the name, zinc-tannin process.

32. **Strength of Timber.**—The properties and strength of different pieces of timber, classed as belonging to the same species, are very variable. Names differ in different parts of the country. The denser and heavier timber is generally the stronger, and seasoned is stronger than green lumber. Prudence would dictate that structures should be designed for the strength of green or moderately seasoned timber of average quality. As the common woods have a comparatively low resistance to compression across the grain, particular attention should be paid to providing sufficient bearing area where a strut or post abuts on the side of another timber. An indentation of the wood destroys the fibre and increases the liability to decay, if the timber is exposed to the weather, especially under the continued working produced by moving loads.

The average breaking stresses of the common woods may be stated as follows, in *thousands* of pounds per square inch.

	Tension		Compression		Bending or Transverse		
	With grain	Across do.	With grain	Across do.	Extreme fibre stress	Modulus of elasticity	Shear with grain
White Oak	10	2	7	2	6	1,100,000	.8
Southern Long Leaf or Georgia Pine	12	.6	8	1.4	7	1,600,000	.6
Douglas or Oregon Fir or Pine	12		8	1.2	6.5	1,400,000	.6
Northern or Short Leaf Yellow Pine	9	.5	6	1	6	1,200,000	.4
Norway Pine	8		6	.8	4	1,200,000	
Spruce	8	.5	6	.7	4	1,200,000	.4
White Pine	7	.5	5.5	.8	4	1,000,000	.4
California Redwood	7			.8	4.5	700,000	.4

Timber has no well-defined yield point.

33. **Timber Specifications.**—The following is a specification for bridge or trestle timber:—All timber shall be of the best quality of white pine, long leaf yellow pine, or white oak, sawed true to size and out of wind (*i. e.*, without twist). It shall be free from sap wood, except in sticks having a depth of 16 inches or upward, when one inch of sap will be allowed on two corners. It shall be free from wind shakes, large knots, loose or rotten knots, wormholes, decayed wood, or other defects that will impair its strength or durability.

Sometimes sap wood is not proscribed, and, in that case, in order that bark may not show on any corner, the requirement is introduced, no *wane*.

Timber sawed from dead trees is very deficient in strength.

34. **Iron and Carbon.**—From the standpoint of the person who uses cast-iron, wrought iron and steel, these materials differ from one another in physical qualities on account of the different percentages of carbon in combination with the iron. The proportion of this ingredient may range from perhaps *five* per cent. to zero, although a portion of the carbon may be replaced by some other element.

35. **Cast-Iron.**—Cast-iron contains the largest percentage of carbon, from two to five per cent., which it gets from the fuel. The ore, fuel, and limestone, the latter added as a flux for the earthy ingredients, are introduced together at the top of a blast furnace, and the molten iron is drawn off at

the bottom, and run into grooves in the sand, the process being continuous. This product is known as pig or cast iron. Although the slag floats on the melted iron, the cast pig, which has taken up from the burning fuel all the carbon it has an affinity for, contains some slag and other impurities. When broken, it is seen to be crystalline in appearance, and it differs in grade from white to gray cast-iron, according to the temperature of the furnace and abundance of fuel. Gray cast-iron is more fluid when melted than white iron, but it requires a higher temperature for its fusion. Gray cast-iron contains *one* per cent., and sometimes less, of carbon in chemical combination with the iron, and from *one* to *three* or *four* per cent. of carbon, in the state of graphite, in mechanical mixture; while white cast-iron is a chemical compound of iron with from *two* to *four* per cent. of carbon. The graphite can be brushed from a freshly broken surface of gray iron, and can have no effect on the metal, except to diminish its strength by preventing a complete union of particles.

The melting point falls, and the hardness and brittleness increase, with the increase of carbon in chemical combination. White cast-iron is of no practical use in itself, but is used in making wrought iron and steel. Gray iron is used in the arts, although its brittleness often renders it objectionable. The softest grade is probably the most fluid, and, since it flows well into the mold, is used for making thin castings, as in ornamental cast-iron work, where strength is not required.

An intermediate grade can be converted from white cast-iron to gray iron by fusion and slow cooling, the carbon having time to separate, and from gray to white by fusion and sudden cooling. When such melted metal is run into a mold lined with iron, it is *chilled*, from the surface for a depth of from one-half to three-fourths of an inch, and made intensely hard, as in the treads of car-wheels. Note the parallelism to the hardening of steel. The tenacity of cast-iron may range from 15,000 lbs. to 30,000 lbs. per square inch of section, and its compressive strength from 60,000 lbs. to 100,000 lbs. Its modulus of elasticity is from 17 to 20 millions. So crude a product is improved by successive remeltings; hence old castings can be melted over with advantage.

36. **Wrought Iron.**—Direct processes for making wrought iron or steel from the ore may be employed, but they are wasteful of iron. Commonly, white pig iron is melted in a cupola furnace and then run into a Bessemer converter or a reverberatory furnace for treatment, which consists principally in the exposure to currents of air to burn out the excess of carbon. Other impurities such as silicon, sulphur and phosphorus may at the same time and by the same means be reduced in amount or burned out.

To explain why wrought iron bars are fibrous:—In the puddling furnace the surface of the melted pig-iron is exposed to a blast rich in oxygen, and the puddler stirs the mass to hasten the burning out of the carbon. As the carbon is removed, the melting point rises, and the iron becomes thick or pasty. Cast-iron does not take on this intermediate state between fluid and solid, but wrought iron does, and hence can be *welded.* The semi-fluid iron is collected into a lump by the puddler and withdrawn from the furnace. It is then much like a sponge; the particles of wrought iron have adhered to one another, but each particle of iron is more or less coated with a thin film of slag and oxide, as water is spread through the pores of a partly dry sponge.

The lump of iron is put into a squeezer, and the fluid slag and oxide drip out as water does from a squeezed sponge. But, as it is impracticable to squeeze a sponge perfectly dry, so it is impracticable to squeeze all the impurities out from among the particles of metallic iron. In the subsequent processes of rolling and re-rolling, each globule of iron is elongated, but the slag and oxide are still there; so that the rolled bar consists of a collection of threads of iron, the adhesion of which to each other is not so great as the strength of the threads.

If the surface of an iron bar is planed smooth and then etched with acid, the metal is dissolved from the surface and the black lines of impurities are left distinctly visible.

That wrought iron is fibrous is then an accident of the process of manufacture, and does not add to its strength. If these impurities had not been in the iron when it was rolled out, it would have been more homogeneous and stronger.

The fibrous fracture of a bar which is nicked on one side and broken by bending is not especially indicative of toughness; for soft steel is tough and ductile without being fibrous.

If the iron first rolled is twice cut up into pieces, piled, reheated and rolled, it makes double refined iron, the grade used for bridges, and superior to single refined iron, or merchant bars. The tensile strength of the former is about 50,000 lbs., varying with the size of the bars; that is, the more work done in rolling, the stronger the iron. The compressive strength is rather low, owing to the ductility of the metal, a pressure of from 36,000 lbs. to 40,000 lbs. per square inch causing decided lateral flow or bending. $E = 28$ or 29,000,000.

Latterly, the manufacture of what is known as soft steel, or homogeneous metal, has been brought to such perfection that steel competes in price with wrought iron and has largely driven it out of the market.

37. **Steel.**—Steel is made from pig-iron by the Bessemer, open-hearth, and other processes, all of which have for their main object the burning out of the carbon, either completely or very nearly. The process is a comparatively rapid one, and several tons are treated at once. The heat generated by the union of the carbon with the oxygen of the air is sufficient to keep the mass fluid, although the melting point rises.

If the product, when practically freed from carbon, is run into molds, and the resulting ingots are rolled out into bars and plates, the material is known as soft or mild steel, ingot iron or homogeneous metal, but is only iron freed from carbon; it is fine grained, tough and ductile, purer and hence stronger than wrought iron. Such a rolled product is not fibrous, like puddled iron, but it is practically the same material, of a better quality. It will weld, but will not harden and temper.

While the reduction of the amount of carbon in the metal under treatment in the furnace or converter may be stopped at a certain desired point, the carbon may be all removed, and then the proper amount may be added to the charge to produce steel with a certain percentage of carbon. By this means the character of the product is better assured. The carbon is

often added in the form of a certain iron, previously melted, of definite composition, known as *spiegel-eisen*. It also contains manganese, which is beneficial in helping to eliminate sulphur. The product of the Bessemer process, very extensively used for steel rails, is not usually so uniform in character as steel made by the open hearth process, where time is afforded for testing the metal and then bringing it to the desired grade.

A very small amount of sulphur in steel or iron will render it red-short, that is brittle and liable to check and crack when hot enough to roll and forge. Phosphorus renders the metal cold-short, or brittle at ordinary temperatures. Current specifications will show how very little of either is tolerated in the best steel.

Since steel is iron with a very small percentage of carbon, it may be made by melting wrought-iron with carbon in a crucible; when the iron takes up carbon, and the product is known as crucible steel. This steel, when rolled, sheared up and rolled again, is shear steel, from which cutting tools are made. Such steel contains from 0.5 to 1.5 per cent. of carbon.

The common soft steel which is used for tension members of bridges, and for pieces exposed to violent use, shocks and vibrations, does not probably contain more than 0.10 to 0.12 of one per cent. of carbon. Steel for compression members, known as medium steel, will contain from 0.12 to 0.24 or 0.36 of one per cent. Thus the carbon in structural steel will range from *zero* to *one-third* of one per cent., a range below the 0.5 of one per cent., for steel which hardens and tempers. Practically the steel used for structures is nothing more than a better grade of iron under a different name.

Steel, properly so-called, will harden and temper, and will not weld; but the use of the electric arc permits the ends of two pieces to be melted together. As the percentage of carbon falls off, steel loses the property of hardening and tempering, and takes on the property of welding. The chilling of a certain grade of cast-iron is analogous to the hardening ot steel. The tensile strength of the softest steel runs from 50,000 lbs. to 60,000 lbs. per sq. inch. This steel is also used for boilers. The next or medium grade of steel will

run from 60,000 lbs. to 70,000 lbs., and the steel which is used for columns may resist at failure from 70,000 lbs. to 80,000 lbs. per sq. inch. Tool steel will go higher. E = 28 to 30,000,000.

The tensile strength of cast-iron, wrought iron and medium steel may be roughly represented by the series 1, 2 and 3, corresponding to 25,000, 50,000 and 75,000 pounds per square inch.

Articles cast from steel are taking the place of iron castings where great strength and toughness are required. They are superior to and cheaper than common forged work. § 47.

To recapitulate:—Wrought or puddled iron is fibrous by reason of the way in which it is made; soft steel is fine grained; the hardest steel is crystalline; cast-iron is coarsely granular.

Cast-iron contains from two to five per cent. of carbon, a part in chemical combination, the rest in mechanical mixture as graphite. Cast-iron contains the most carbon; steel, such as is used for machine and other tools, has a medium anount of carbon; the soft, structural steel has very little, and some of it, practically, none at all; and wrought iron has no carbon.

38. **Malleable Iron: Case-hardened Iron.**—There are two other products which may well be mentioned, and which will be seen to unite or fit in between the three already described. The first is what is known as "malleable cast-iron" or malleable iron.

Small articles, thin and of irregular shapes, which may be more readily cast than forged or fashioned by a machine, and which need not be very strong, are made of cast-iron, and then imbedded in a substance rich in oxygen, as, for instance, powdered red hematite iron ore, sealed up in an iron box, and heated to a high temperature for some time. The oxygen abstracts the carbon from the metal to a slight depth, converting the exterior into soft iron, malleable iron, with an increase of strength and diminution of brittleness.

The second product is case-hardened iron. An article fashioned of wrought iron or soft steel is buried in powdered charcoal and heated. The exterior absorbs carbon and is

converted into high steel, which will better resist wear and violence than will soft iron. The Harvey process for hardening the exterior of steel armor plates is of a similar nature.

39. **Work of Elongation.**—It is seen from the diagram, Fig. 1, that the resistance of the metal per square inch increases as the bar draws out and diminishes in section under tension, as shown by the dotted curve, although the total resistance grows less near the close of the test, as shown by the full line. As a small increase in the amount of carbon diminishes the elongation and reduction of area, it is possible that the carbon affects the apparent ultimate strength in this manner (since such strength is computed on the square inch of original section), and not by actually raising the resisting power of the metal.

Since the measure of the work done in stretching a bar is the product of one-half the force by the stretch, if the yield point has not been passed, and, for values beyond that point, is the area below the curve in the diagram, limited by the ordinate representing the maximum force,—the comparative ability of a material to resist live load, shock and vibration is indicated by this area. A mild steel of moderate strength may thus have greater value than a higher carbon steel of much greater tensile strength.

Such a measure of work done to produce fracture is aimed at in the following specification:

In tensile tests of steel, round sample bars, not less than $\frac{3}{4}$ in. in diameter, and not less than 12 diameters long between the jaws of the testing machine, shall show a percentage of elongation not less than 1,200,000 ÷ the tensile strength per square inch, and a percentage of reduction of area not less than double the same ratio. The elongation shall be measured after breaking, on an original length of ten times the diameter of the piece, in which length must occur the curve of reduction from stretch on both sides of the point of the fracture.

40. **Classification.**—La Société Cockerill has proposed the following classification for steel:—

Steel.	Carbon, %.	Breaking Strength.	Elongation, %.	
Extra Soft..	0.05 to 0.20	57,000 to 70,000	27 to 20	Welds but does not temper.
Soft..........	0.20 to 0.35	70,000 to 85,000	20 to 15	Welds with difficulty; does not temper.
Hard........	0.35 to 0.50	85,000 to 100,000	15 to 10	Does not weld. Tempers.
Extra Hard	0.50 to 0.65	100,000 to 115,000	10 to 5	Does not weld. Tempers hard.

The first and second grades would more commonly be called *soft* and *medium*.

From recent experiments it appears that wrought iron, or steel almost free from carbon, can be hardened, if heated hot

enough before sudden cooling, and that a great increase in strength is thus developed. This fact tends to confirm the view that increased strength is not due to carbon percentage but to diminished lateral contraction under longitudinal tension. See § 200.

Iron may also be increased in softness by very carefully annealing.

41. Specifications for Cast-Iron.—All castings must be of tough, gray iron, free from cold shuts or injurious blow holes, true to form and thickness, and of a workmanlike finish. Sample pieces, one inch square, cast from the same heat of metal in sand molds, shall be capable of sustaining, on a clear span of 4 ft. 6 in., a central load of 500 lbs., when tested in the rough bar. A blow from a hammer shall produce an indentation on a rectangular edge of the casting without flaking the metal.

Cast-iron for water pipes is usually required to have from 16,000 to 18,000 lbs. per sq. in. tensile strength, and to be soft enough to admit readily of tapping and cutting a thread.

42. Specifications for Wrought Iron.—Wrought iron has been to a great degree displaced by mild or soft steel. The following specifications for wrought iron agree very well with recent practice:—

All wrought iron must be tough, fibrous and uniform in character. It shall have a yield point of not less than 26,000 lbs. per sq. in. Finished bars must be thoroughly welded during rolling, and be free from injurious seams, blisters, buckles, cinder-spots, or imperfect edges.

For all tension members, the muck bars shall be rolled into flats, and again cut, piled and rolled into finished sizes. They shall stand the following tests:—Full-sized pieces of flat, round or square iron, not over $4\frac{1}{2}$ square inches in sectional area, shall have an ultimate strength of 50,000 lbs. per sq. in., and stretch $12\frac{1}{2}$ per cent. in their whole length.

Bars of larger section than $4\frac{1}{2}$ square inches, when tested in the usual way will be allowed a reduction of 1,000 lbs. for each additional square inch, down to a minimum of 46,000 lbs. per square inch.

When tested in specimens, of uniform section of at least one-half a square inch in a length of ten inches, taken from tension members rolled to a section not more than $4\frac{1}{2}$ square

inches, the iron shall show an ultimate strength of 52,000 lbs., and stretch 18 per cent. in a measured distance of eight inches. Specimens from bars larger than $4\frac{1}{2}$ square inches will be allowed a reduction of 500 lbs. for each additional square inch of section, down to minimum of 50,000 lbs.

The same sized specimen taken from angle and other shaped iron, or from plate iron, shall have an ultimate strength of 50,000 lbs., and elongate 15 per cent. in eight inches.

All iron for tension members must bend cold for about 90°, to a curve of diameter not over twice the thickness of the piece, without cracking; and at least one sample in three must bend 180° to this curve without cracking. When nicked on one side and bent from a blow with a sledge, the fracture must be nearly all fibrous, showing but few crystalline specks. Specimens from angle, plate and shaped iron must stand bending cold through 90°, to a curve of diameter not over three times the thickness of the piece, without cracking. When nicked and bent, the fracture of the specimen must be mostly fibrous.

43. Specifications for Steel.—The following specifications have been used for structural steel:—

Steel for rivets and eyebars shall contain not more than one quarter of one per cent. of carbon, and less than one tenth of one per cent. of phosphorus. A sample bar $\frac{3}{4}$ in. in diameter, when tested in a lever machine, shall have a yield point of not less than 40,000 lbs. per square inch, and an ultimate strength of not less than 70,000 lbs. per square inch; it shall elongate at least 18 per cent. in a length of eight inches, and shall show a reduction of area of at least 45 per cent. at the point of fracture. In full-sized bars this steel shall have a yield point of at least 35,000 lbs., and an ultimate strength of at least 65,000 lbs. per square inch; it shall elongate 10 per cent. before breaking, and for stresses less than 30,000 lbs. per square inch, it shall show a modulus of elasticity between 28,000,000 and 30,000,000.

A sample bar $\frac{3}{4}$ inch in diameter shall bend 180° cold, and be set back upon itself without showing crack or flaw.

Steel used in compression members shall not contain more than one-tenth of one per cent. of phosphorus. When

tested in tension, a sample bar ¾ in. in diameter shall have an ultimate strength of not less than 80,000 lbs. per square inch, and a yield point of not less than 50,000 lbs; it shall elongate at least 15 per cent. in eight inches, and show a reduction of area of at least 30 per cent. at the point of fracture. It shall also bend 180° cold around its own diameter without showing crack or flaw. It shall be incapable of tempering. The following are examples of steel, not annealed:—

Yield Point.	Ultimate Strength.	Elongation in 8 in.	Reduced Area.	Carbon.	Manganese.	Phosphorus.	
53,140 lbs.	83,680 lbs.	20.75%	37.78%	0.27	0.83	.067	of one per ceut
51,190 lbs.	84,440 lbs.	18.75%	33.23%	0.26	0.79	.067	

For structural and architectural work the following requirements are reasonable:

Rolled steel shall not contain more than 0.04 per cent. of phosphorus. It shall be finished straight and smooth. It shall show by the standard test piece an ultimate tensile strength of 56,000 ± 4,000 lbs. per sq. inch, with an elongation of 25 per cent. in eight inches. The yield point shall in no case be less than 55 per cent. of the maximum tension sustained by the test piece (.55 . 56,000 = 30,800), and it may be indicated by the "drop of beam" of the testing machine. Each melt of finished material shall receive three tests: two tension (one cut from each extreme variation in thickness of metal), and one hot bending. When the chemical analysis shows less than 0.05 per cent. of sulphur, the hot bending test will not be required.

44. **Machinery Steel.**—When much machine work is to be done on a piece, use for forgings mild steel of from 0.20 to 0.25 of one per cent. of carbon, with a yield point of 27,000 lbs. per sq. inch, an ultimate strength of 57,000 lbs. per sq. inch, and an average elongation of 28 per cent. in a length of four diameters.

For the general run of engine forgings use a harder steel of 35,000 lbs. per sq. inch yield point, an ultimate strength of 75,000 lbs., and having 25 per cent. elongation in four diameters.

With precautions, forgings may be made from steel of a still higher grade, for crank and crosshead pins, and parts subjected to severe alternating stresses and wearing action, such steel to show a yield point of 40,000 lbs. per sq. inch, an ultimate strength of 85,000 lbs., and 20 per cent. elongation in four diameters.

If the shape of the piece will allow tempering, the above values may be raised to a yield point of 45,000 lbs., an ultimate strength of from 85,000 to 90,000 lbs., and 23 per cent. elongation.

A small percentage of nickel added to mild steel increases the strength greatly without causing loss of ductility. The hollow shafts for the U. S. S. Brooklyn, 17 inches external diameter, 11 inches internal diameter, 39 feet long, showed 60,775 lbs. yield point, 94,245 lbs. ultimate strength, and 60.5 per cent. reduction of area.

Fluid compressed, oil-tempered steel, containing from 0.4 to 0.45 of one per cent. of carbon, and especially nickel-steel, is suitable for piston-rods of rock-drills, hammer rods, stamp stems, cam shafts, crank and crosshead pins, and pieces subjected to alternating stresses of tension and compression, or to either kind frequently repeated. The yield point will be from 50,000 to 60,000 lbs.

Steel castings are now successfully made, although formerly much difficulty was experienced in securing soundness. Some have shown, before annealing, a tensile strength of 90,000 lbs. per square inch, an elongation of 22%, and a reduction of area of from 30 to 40%.

45. **Chemical Specifications and Manipulation.**—It is the opinion of some engineers that neither the chemical constitution nor the mechanical processes of manufacture should be specified, in calling for a certain grade of steel, but only breaking strength, yield point, elongation and reduction of area. The following, however, are examples of requirements in some specifications:—

Acid open-hearth and Bessemer steel limits for phosphorus may range from 0.06 to 0.1 of one per cent., and for sulphur one-half of these percentages. Basic open-hearth steel may have from 0.02 to 0.06 of one per cent. of phosphorus, and from 0.02 to 0.04 of one per cent. of sulphur.

Structural Steel:—To be acid or basic open-hearth. Acid steel to have phosphorus below 0.06%; basic, below 0.04%. Sulphur to be below 0.10; silicon, below 0.10; manganese, below 0.65. To show 56,000 to 64,000 lbs. ultimate strength; ratio of yield point to ultimate strength, 50 to 63%; elongation, 27 to 22%; and reduction of area 50 to 40%.

Another specification for structural steel for railroad bridges says:—

All raw material used in the manufacture of steel ingots shall be chemically within the Bessemer limit of phosphorus 0.10 of one per cent., sulphur 0.05, copper 0.40.

All ingots shall be cast from steel melted in an acid-lined open-hearth furnace. No single ingot or casting shall exceed 15,000 lbs. in weight, in order to avoid extreme segregation. All ingots must be bottom cast, and no ingot shall be disturbed or removed from the position in which it is cast until it is sufficiently solidified to obviate "bleeding."

Finished rolled steel shall show under analysis not more than 0.08% phosphorus, 0.04 sulphur, 0.45 manganese, 0.20 copper.

It shall be straight, well finished in the rolling, full to dimensions, and free from lamination, buckles, and surface, edge or other defects. It shall show in test pieces, for plates and shapes, an ultimate strength of not less than 58,000 lbs., nor more than 65,000 lbs., a yield point of 38,000 lbs., an elongation, for plates under 36 inches wide, of 26%, and for plates over 36 inches wide, of 24%, and a reduction of area of 50%.

Rivet rod shall have an ultimate strength of between 50,000 and 54,000 lbs., with a reduction of area of 60%.

46. Punching and Drifting Tests.—Punching test, applied to steel plate, for stand pipe or boiler work, having 55,000 to 65,000 lbs. tensile strength, a yield point of not less than 30,000 lbs., and in 3/8 inch plate an elongation of 25% longitudinally, or 22% transversely:—A piece 1 3/4 in. by 10 in. shall permit of punching a row of not less than eight 3/4 inch holes, 1 1/4 inches centre to centre, without crack. Drifting test:—A piece 3 in. by not less than 5 in. shall undergo the punching of not less than two 3/4 inch holes, ? in. centre to centre, and 1 1/2 inches from edges, said holes to be then enlarged cold by a sledge and drift-pin to at least 1 1/4 in. diameter, without cracking. The steel must stand cold hammering or scarfing at lap joints without cracking. It is hardly necessary to add, that it shall bend cold 180° on itself without crack.

47. Effect of Shearing and Punching.—Steel, other than of the softest grade, is thought to be weakened by shearing and punching, by the development of minute cracks, which, however, do not extend to any distance. To remove this weakened portion the edges of sheets are often planed after shearing, and punched holes are reamed. The specification for *medium* steel will read:—

All sheared edges of plates and angles shall be planed off to a depth of one-fourth of an inch. All punched holes shall be reamed to a diameter one-eighth inch larger, so as to remove all the sheared surface of the metal.

Sharp re-entrant corners are not allowed in good practice.

48. Stone.—Stone for building differs very much in composition and quality. The igneous or primary rocks are generally satisfactory in the matter of strength, but their hardness makes them expensive to fashion to other than simple forms. Stratified stones, like the limestones and sandstones, are found in all grades, from the hardest and most durable to the softest and most perishable. The only sure test of the ability of a building stone to resist climatic changes, to stand the weather, is the lapse of time. Artificial freezing and thawing of a small specimen, frequently repeated, will give indication as to durability.

Sound hard stones, like granite, gneiss, clay-slate, marble, compact limestone, and the better grades of sandstone, are sufficiently strong to carry any loads brought upon them in ordinary buildings. In exceptional cases, special investigation should be made.

Stratified stones should be laid on their natural bed, that is, so that the pressure shall come practically perpendicular to the layers. They are much stronger in such a position, and the moisture which porous stones absorb from the rain can readily dry out. If the stones are set on edge, the moisture is retained and, in the winter season, tends to dislodge fragments by the expansive force exerted when it freezes. Some sandstone facings rapidly deteriorate from this cause. Crystals of iron pyrites occur in some sandstones and unfit them for use in the face of walls. The discoloration resulting from

their oxidation, and the local breaking of the stone from the swelling are objectionable.

49. Masonry.—Most masonry consists of regularly coursed stones on the face, with a backing of irregular shaped stones behind, Stones cut to regular form and laid in courses make ashlar masonry, if the stones are large and the courses continuous. When the stones are smaller, and the courses not entirely continuous, or sometimes quite irregular, although the faces are still rectangular, the descriptive name is somewhat uncertain, as block-in-course, random range, etc., down to coursed rubble, where the end joints of the stones are not perpendicular to the beds. Rubble masonry denotes that class where the stones are of irregular shape, and fitted together without cutting. If the face of a stone is left as it comes from the quarry, the work is called quarry-faced or rock-faced. The kind of masonry depends upon the beds and joints. Walls of stone buildings have only a more or less thin facing of stone, the body of the wall being of brick. The stone facing should be well anchored to the brick work by iron straps.

50. Specifications.—The following specifications will indicate how good work is described. Beginning with the portion under ground:—

All foundation courses shall be built of rubble masonry, with selected, large flat stones not less than twelve inches thick, nor of less superficial surface than fifteen square feet. The foundation shall be brought to a level bed at the footing course, selected stones being used to give the six-inch offset required. The spaces between stones when laid close, shall at no place be over six inches in width, and these spaces shall be filled with small stones and spalls, flushed in cement mortar and well grouted.

For rubble masonry above the foundation courses:—The stones used shall be quarried or split stones, hard, sound, and as nearly rectangular as possible, of good, flat beds and builds, and, unless used for trimming or closers, not less than six inches in thickness when laid on the largest face, and at right angles to the face of the wall. At least one-third of the stones shall be headers and extend back from the face not less

than three feet or clear through the wall. A proper alternation of headers and stretchers shall be used in order to secure thorough bond. The stones shall be laid in full beds of cement mortar, with joints completely filled. The joints shall not exceed one inch in thickness, and the courses are to be properly leveled up. The stone shall be washed clean immediately before being laid. No dressing nor tooling will be allowed on any stone after it is in place.

51. **Ashlar Masonry.**—The masonry above the footing course shall consist of quarry-faced ashlar, laid in horizontal courses, having parallel beds and vertical joints. The courses shall be not less than 12 inches nor more than 30 inches in thickness, decreasing in thickness regularly from the bottom to the top of the wall, and shall be laid flush in cement mortar. Each course shall be thoroughly grouted before the succeeding one is laid. The stones shall be alternately headers and stretchers, and every header shall be laid immediately above a stretcher in the underlying course. All stones in every course shall bond at least 12 inches with those in the preceding course. The stretchers shall be not less than 3 feet nor more than 6 feet in length, and not less than 2 feet in width, nor less in width than one and a half times their depth. The headers shall be not less than $3\frac{1}{2}$ nor more than $4\frac{1}{2}$ ft. in length, where the thickness of the wall will permit, and not less than $1\frac{1}{2}$ ft. in width, nor less in width than they are in depth of course.

Every stone shall be laid on its natural bed, which bed shall be well dressed to a plane surface and made as large as the stone will permit. The beds and sides of the stones shall be cut before being placed on the work, so that the mortar joints on the face shall not exceed one-half inch. No hammering on a stone shall be allowed after it is set; any inequalities shall be pointed off. The vertical joints shall be cut to the same joint for a distance of not less than 12 inches from the face. All corners and batter lines shall be run with a chisel draft two inches wide on each face.

The backing shall be of good-sized well-shaped stones, laid so as to break joints and thoroughly bond the work in all directions, and leave no spaces over 6 inches in width. These

spaces shall be filled with small stones and spalls, flushed in cement mortar and well grouted.

All bridge seats and tops of walls shall be finished with a dressed coping course, the joints in which shall not exceed one-fourth inch. Each stone shall be not less than 4 feet long, and all shall be 12 inches thick. The bridge seats shall have four, and the coping two $1\frac{1}{2} \times 1\frac{1}{2}$ inch wrought iron dowels to each stone.

No stone shall be shifted in its mortar bed by bars, and no movement of the stone laterally after being placed upon the wall will be permitted. Each stone shall be lowered to place dry, and shall then be raised, a bed of mortar spread to receive it, and the stone lowered to place.

Every surface to which mortar is to be applied shall be freed from dust and dirt and thoroughly sprinkled just before the mortar is spread.

Not more than three courses of the abutment shall be unfinished at a time, and the backing shall be carried up with the facing, but never in advance of it. Each course of masonry shall be grouted as laid with a mixture of two parts sand, one part cement and no more water than is necessary to give the required fluidity. The grout shall be worked into the vertical joints thoroughly with suitable flat iron blades, until all air is expelled and the joints completely filled.

All outside joints of the masonry shall be raked out to the depth of one inch and neatly pointed with a mortar of one part Portland cement and one part sand.

All mortar used in the masonry shall consist of two parts by measure of sand and one part of American cement equal to the best quality of freshly burned Rosendale cement, to which shall be added only water enough to form a paste, stiff enough to handle with a trowel. The mortar shall be mixed in small quantities, as required for use, and shall be used before it has taken an initial set. The sand shall be sharp and clean, free from loam and clay.

52. **Brick Clay.**—Clay may be roughly stated to be silicate of alumina. There are different grades of clay, from some of which china and porcelain are made, from others fire bricks, and from others common, or building bricks. The last

clays are the most easily fusible. Good brick clay, thoroughly burned, will yield hard, well shaped bricks. A too fusible clay will not allow sufficient burning, and hence the bricks will be comparatively soft. Lime in the clay lowers the fusing point, and the presence of lumps of lime in the bricks is a serious matter, as such lumps, when the bricks are wetted and laid in the wall, will slake, swell, and break the bricks.

53. **Bricks.**—A good brick should be straight and sharp-edged, reasonably homogeneous when broken, dense and heavy. Two bricks struck together should give a ringing sound. Bricks which have a smooth exterior have been pressed after molding, and are more expensive. Some bricks, such as are used for paving, are made from ground shale.

Soft, under-burned bricks are very porous, absorb much water, and cannot be used on the outside of a wall, especially near the ground line, for they soon disintegrate from freezing. Hard-burned bricks are very strong and satisfactory in any place; they can safely carry six or eight tons to the square foot.

The red color of common bricks is an accidental characteristic, due to iron in the clay. Such bricks are redder, the harder they are burned, finally, in some cases, turning blue. The cream-colored bricks with no iron may be just as strong and are common in some sections.

The builder usually lays the face of the wall with the best bricks, and the interior may be filled with the softer bricks, and even with bats, if permitted. The workman is not likely to fill the end joints with mortar unless an inspector insists on it. A shove or push joint is sometimes specified to cover this point.

Bricks differ much in size in different parts of the country. The thickness of walls and the size of stone trimmings are to be adapted to the width and thickness of courses of brick. A course of headers is usually laid after from four to six courses of stretchers; but sometimes headers and stretchers alternate in every course.

54. **Lime.**—Lime for use in ordinary masonry and brickwork is made by burning limestone, or calcium carbonate, and thus driving off by a high heat the carbon dioxide and

such water as the stone contains. There remains the quick-lime of commerce, in lumps and powder. This quick-lime has a great affinity for water and rapidly takes it up when offered, swelling greatly and falling apart, or slaking, into a fine, dry, white powder, with an evolution of much heat, due to the combination of the lime with the water. The use of more water produces a paste, and the addition of sand, which should be silicious, sharp in grain and clean, makes lime mortar. The sand is used, partly for economy, partly to diminish the tendency to crack when the mortar dries and hardens, and partly to increase the crushing strength. The proportion is usually 2 or $2\frac{1}{2}$ parts by measure of sand to one of slaked lime in paste, or 5 to 6 parts of sand to one of unslaked lime. As lime tends to air-slake, it should be used when recently burned.

Some limes slake rapidly and completely; other limes have lumps which slake slowly and should be allowed time to combine with the water. It is generally considered that lime mortar improves by standing, and that mortar intended for plastering should be made several days before it is used. Small unslaked fragments in the plaster will swell later and crack the finished surface. The lime paste is sometimes strained to remove such lumps.

Lime mortar hardens by the drying out of part of the water which it contains, and by the slow absorption of carbon dioxide from the air. It thus passes back by degrees to a crystallized calcium carbonate surrounding the particles of sand. Dampness of the mortar is favorable to the attainment of this result, and the mortar in a brick wall which has been kept damp for some time, will harden better than where the wall is dry. Dry, porous bricks absorb rapidly, and almost completely, the water from the mortar, and reduce it to a powder or friable mass which will not harden satisfactorily. Hence bricks should be well wetted before they are laid.

Lime mortar in the interior of a very thick wall may not harden for a long time, if at all, and hence should not be used in such a place. Slaked lime placed under water will not harden, as may be proved by experiment. In

both cases, such inaction is due to the exclusion of the carbon dioxide. Lime mortar should never be used in wet foundations.

Plaster for interior walls is lime mortar. Hair is added to the mortar for the first coat, so that the portion which is forced through the spaces between the laths and is *clinched* at the back may have sufficient tenacity to hold the plaster on the walls and ceiling.

55. Natural Cement.—*Natural* cement is made by burning almost to vitrification a rock which contains lime, silica and alumina, that is, one which may be considered a mixture of a limestone and a clay rock. The carbon dioxide, moisture and water of crystallization are expelled by burning. The hard fragments must then be ground to powder, the finer the better. If the rock contains the several ingredients in proper proportion, upon the addition of water to the powder a reaction promptly takes place, and the double silicate of lime and alumina is formed, with a certain amount of water of crystallization. The fine grinding is necessary for a thorough mingling of the particles. The hardening which indicates the above reaction is called *setting*, and, in cements which contain the proper ingredients, begins in from a few minutes to half an hour. Those cements which contain an excess of lime set more slowly. The hardening, the tensile and compressive strength, increase rapidly at first, and at a decreasing rate for months.

As access of air is not required for the setting of cement, the reaction taking place when water is added to the dry powder, cement mortar is used invariably under water and in wet places. It makes stronger work than lime mortar, and is generally used by engineers for stone masonry. Its greater cost than that of lime is due to the necessity of grinding the hard slag; while lime falls to powder when wet. The proportion of sand is 1, 2, or 3 to one of cement, according to the strength desired, 2 to 1 being a common ratio for good work. The sand and cement are mixed dry and then wetted, in small quantities, to be used at once.

The slower setting cement mortars are likely to show a greater strength, some months or years after use, than do the

quick setting ones, which attain considerable strength very soon, but afterwards gain but little.

56. **Portland Cement.**—If the statement made as to the composition of cement is correct, it should be possible to make a mixture of chalk, lime or marl and clay in proper proportions for cement, and the product ought to be more uniform in composition and characteristics than that from the natural rock. Such is the case; and the artificial cement, obtained by carefully mixing balls of clay and marl or lime, burning the balls nearly to vitrification and finely grinding, is known as Portland cement, an article superior to the natural cement, such as the Rosendale, Akron or Louisville brands. Much Portland cement is imported from Europe.

The addition of brick-dust from well burned bricks to lime mortar will make the latter act somewhat like cement, or become hydraulic, as it is called. Volcanic earth has been used in the same way.

57. **Concrete.**—Concrete is a mixture of cement mortar (cement and sand) with gravel and broken stone, the materials being so proportioned and thoroughly mixed that the gravel fills the spaces among the broken stone; the sand fills the spaces in the gravel; and the cement is rather more than sufficient to fill the interstices of the sand, coating all, and cementing the mass into a solid which possesses in time as much strength as many rocks. It is used in foundations, floors, walls, and for complete structures. The broken stone is usually required to be small enough to pass through a 2 in. or 2½ in. ring. The stone is sometimes omitted.

The concrete should not be made very soft and wet, but rather *mealy*, and should be deposited in place as soon as mixed, in layers from six to ten inches thick, and rammed till well consolidated, indicated by water slightly flushing to the surface.

The proportions for mixing can be ascertained by filling a box or barrel with broken stone shaken down, and counting the buckets of water required to fill the spaces; then empty the barrel, put in the above number of buckets of gravel and count the buckets of water needed to fill the interstices of the gravel; repeat the operation with that number of buckets of

sand, and use an amount of cement a little more than sufficient to fill the spaces in the sand. If the gravel is sandy, screen it before using, in order to keep the proportions true.

A common rule is, one part cement, two parts sand and five parts of broken stone or pebbles by measure. In another case, $5\frac{1}{2}$ c. ft. cement, 7 c. ft. sand and 27 c. ft. broken stone made a cubic yard of concrete. A good and easily remembered rule is—one part cement, 2 parts sand, 3 parts coarse gravel and 4 parts broken stone.

58. **Cement Specifications.**—The following specifications for cements are reasonable, and the requirements are often exceeded:—

Natural cement, when tested with a No. 50 sieve, shall not leave a residue of more than 4% by weight; on a No. 100 sieve, not more than 25%; and on a No. 200 sieve, not more than 50% by weight.

A pat of the same, $\frac{1}{2}$ in. thick, at 60° to 70° F., shall take an initial set in not less than 10 minutes, and shall set hard in not less that 30 minutes.

Neat cement briquettes, one day in water after hard set, shall show a tensile strength of at least 75 lbs. per sq. inch. After one day in air and six in water, 150 lbs.; 28 days, 250 lbs. Mixture of one part cement, one standard crushed quartz, 7 day test, 75 lbs. Briquettes from mixing box, two sand to one cement, 40 lbs.

For Portland cement, the residue as above shall be not more than 1% on No. 50 sieve, 15% on No. 100, and 40% on No. 200. A pat of $\frac{1}{2}$ in. thickness in a stiff paste shall require at least 30 minutes for an initial set.

Neat cement briquettes, 24 hours in water after hard set, shall develop 125 lbs. tensile strength per sq. inch. After one day in air and six in water, 350 lbs.; 28 days, 450 lbs. Cement mortar, with 3 parts standard crushed quartz to 1 cement, one day in air and six in water, shall show 125 lbs. Portland cement briquettes from the mixing board, two sand to one cement, one day in air and six in water, shall show 150 lbs.

The three sieves shall be made of wire cloth, No. 25, No. 40 and No. 40 wire respectively, Stubbs gauge, and having 2,500, 10,000 and 40,000 meshes per sq. inch.

The modulus of elasticity for concrete is about 700,000; for neat cement, about 3,000,000. The cohesion of iron and concrete is about 600 lbs. per sq. inch; of stone and cement about 15 lbs. per sq. inch. Brick masonry will fracture through the bricks rather than the joints if laid in thoroughly good cement mortar.

The modulus of rupture, or f for breaking weight of a beam, may be taken as about twice the tensile strength of the mortar used.

59. Masonry Laid in Winter.—Civil engineers generally discontinue masonry construction as soon as freezing weather is likely to occur; but contractors in cities frequently carry up brick walls in lime mortar, although a temperature of 0° F. may be experienced. In such a case the lime should be slaked with hot water, the bricks should be heated and laid in hot mortar. A thaw during erection is injurious to a wall built in winter, but continuous freezing is not deemed harmful. A man can lay only about half as many bricks at such a time.

Salt is often dissolved in the water with which cement mortar is mixed when it is to be used in freezing weather. Much salt will weaken the mortar.

Natural cement concrete will disintegrate for a short distance below the surface, if exposed to a northern winter; but Portland cement is unimpaired by the action of frost, if well laid.

CHAPTER III.

BEAMS.

60. Beams: Reactions.—A beam may be defined to be a piece of a structure, or the structure itself as a whole, subjected to transverse forces and bent by them. If the given forces do not act at right angles to the axis or centre line of the piece, their components in the direction of the axis cause tension or compression, to be found separately and provided for; the normal or transverse components alone produce the beam action or bending.

As all trusses are skeleton beams, the same general principles apply to their analysis, and a careful study of beams will throw much light on truss action.

Certain forces are usually given in amount and location on a beam or assumed. Such are the loads, concentrated at points or distributed over given distances, and due to the action of gravity; the pressure arising from wind, water or earth; or the action of other abutting pieces.

It is necessary, in the first place, to satisfy the requirements of equilibrium, that the sum of the transverse forces shall equal zero and that the sum of their moments about any point shall also equal zero. This result is accomplished by finding the magnitudes and direction of the forces required at certain given points, called the points of support, to produce equilibrium. The supporting forces, or *reactions*, exerted by the points of support *against* the beam, are two or more, except in the rare case where the beam is exactly balanced on one point of support. For cases where the reactions number more than two, see § 122.

61. Beam supported at two points. Reactions.—The simplest and most generally applicable method for finding one of the two unknown reactions is to find the sum of the moments of the given forces about one of the points of support, and to equate this sum with the moment of the other

reaction about the same point of support. Hence, divide the sum of the moments of the given external forces about one of the points of support by the distance between the two points of support, usually called the *span*, to find the reaction at the other point of support. The direction of this reaction is determined by the sign of its moment, as required for equilibrium. The amount of the other reaction is usually obtained by subtracting the one first found from the total given load.

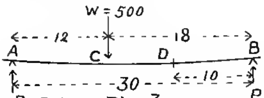


Fig. 3.

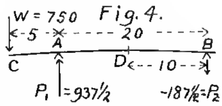


Fig. 4.

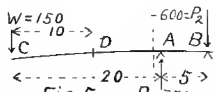


Fig. 5.

Thus, in the three cases sketched, $P_1 = W \frac{C\ B}{A\ B}$; $P_2 = W - P_1$.

Examples.—Fig. 3. If $W = 500$ lbs., $A\ B = 30$ ft., and $B\ C = 18$ ft.; $P_1 = \frac{500 \cdot 18}{30} = 300$ lbs., $P_2 = 500 - 300 = 200$ lbs.

Fig. 4. If $W = 750$ lbs., $A\ B = 20$ ft. and $A\ C = 5$ ft., $P_1 = \frac{750 \cdot 25}{20} = 937\frac{1}{2}$ lbs., and $P_2 = 750 - 937\frac{1}{2} = -187\frac{1}{2}$ lbs.

Fig. 5. If $W = 150$ lbs., $A\ C = 20$ ft. and $A\ B = 5$ ft., $P_1 = \frac{150 \cdot 25}{5} = 750$ lbs., and $P_2 = 150 - 750 = -600$ lbs. Note the magnitude of P_1 and P_2 as compared with W when the distance between P_1 and P_2 is small. Such is often the case when the beam is built into a wall.

Where the load is distributed at a known rate over a certain length of the beam, the resultant load and the distance from its point of application to the point of support may be conveniently used.

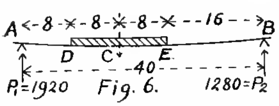


Fig. 6.

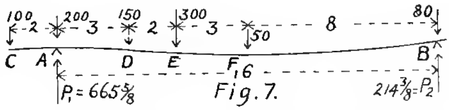


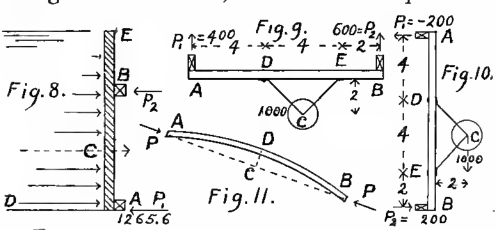
Fig. 7.

Example.—If $A\ B = 40$ ft., $A\ D = 8$ ft., $D\ E = 16$ ft., and the load on D E is 200 lbs. per ft., $W = 3{,}200$ lbs., and $C\ B = 24$ ft. Therefore $P_1 = \frac{3{,}200 \cdot 24}{40} = 1{,}920$ lbs., and $P_2 = 3{,}200 - 1{,}920 = 1{,}280$ lbs.

If several weights are given in position and magnitude, the same process for finding the reactions, or forces exerted by the points of support against the beam is applicable.

Examples.—In Fig. 7, $P_1 = (100 \,.\, 18 + 200 \,.\, 16 + 150 \,.\, 13 + 300 \,.\, 11 + 50 \,.\, 8 + 80 \,.\, 0) \div 16 = 665\frac{5}{8}$ lbs. $P_2 = 880 - 665\frac{5}{8} = 214\frac{3}{8}$ lbs. The work can be checked by taking moments about A to find P_2, the moment $100 \,.\, 2$ then being negative.

If the depth of water against a bulkhead, Fig. 8, is 9 ft., and the distance between A and B, the points of support, is 6 ft., A being at the bottom, the unit water pressure at A will be $9 \times 62.5 = 562.5$ lbs., which may be represented by A D, and at other points will vary with the depth below the surface, or as the ordinates from E A to the inclined line E D. Hence the total pressure E A, for a strip one foot in horizontal width, will be $562.5 \times 9 \div 2 = 2{,}531\frac{1}{4}$ lbs., and the resultant pressure will act at C, distant $\frac{1}{3}$ A E, or 3 ft. from A. $P_2 = 2{,}531\frac{1}{4} \times 3 \div 6 = 1{,}265.6$ lbs., and $P_1 = 2{,}531.2 - 1{,}265.6 = 1{,}265.6$ lbs., a result that might have been anticipated, from the fact that the resultant pressure here passes midway between A and B.



Let 1,000 lbs. be the weight of pulley and shaft attached by a hanger to the points D and E, Fig. 9. Let the beam A B = 10 ft., A D = 4 ft., D E = 4 ft., E B = 2 ft.; and let C be 2 ft. away from the beam. As the beam is horizontal, $P_1 = 1{,}000 \times 4 \div 10 = 400$ lbs.; $P_2 = 1{,}000 - 400 = 600$ lbs., and both act upwards. The 1,000 lbs. at C causes two vertical downward forces on the beam, each 500 lbs., at D and E. There is also compression of 500 lbs. in D E.

When the beam is vertical, Fig. 10, by moments, as before, about B, $P_1 = 1{,}000 \,.\, 2 \div 10 = 200$ lbs. at A acting to the left, being tension or a negative reaction. By moments about A, $P_2 = 1{,}000 \,.\, 2 \div 10 = 200$ lbs. at B, acting to the right. Or $P_1 + P_2 = 0$; $\therefore P_1 = -P_2$. By similar moments, the 1,000 lbs. at C causes two equal and opposite horizontal forces on the beam at D and E, of 500 lbs. each, that at D being tension on the connection, or acting towards the right, and that at E acting in the opposite direction. These two forces make a couple, balanced by the couple P_1 P_2. The weight 1,000 lbs. multiplied by its arm 2 ft. is balanced by the opposing horizontal forces at D and E, 4 ft. apart. There remains a vertical force of 1,000 lbs. in A B, which may all be resisted by the point B, when the compression in D E = 500 lbs. and in E B = 1,000 lbs.; or all by the point A, when the ten-

sion in D E = 500 lbs. and in D A = 1,000 lbs.; or part may be resisted at A and the rest at B, the distribution being uncertain. This longitudinal force may be disregarded in discussing the beam, as may the tension or compression in the hanger arms themselves.

62. Bending Moments.—If an imaginary plane of section is passed through any point in a beam, the sum of the moments of all the external forces on one side of that section, *taken about a point in the section*, must be exactly equal and opposite to the sum of the moments of all the external forces on the other side of that section, taken about the same point. If not, the beam would revolve in the plane of the forces. The moment on the left side of the section tends to make that portion of the beam rotate in one direction about the point of section, and the equal moment on the right side of the section tends to make the right segment rotate in the opposite direction. These two moments cause resistances in the interior of the beam at the section (which stresses will be discussed under *resisting moment*), with the result that the beam is bent to a slight degree. Either resultant moment on one side of a plane of section, about the section, is called the *bending moment* at that point, usually denoted by M, and is considered *positive* when it makes the beam *concave* on the upper side. Ordinary beams, supported at the ends and carrying loads, have positive bending moments.

If upward reactions are positive, weights must be taken as negative and their sign regarded in writing moments.

Examples.—Section at D, Fig. 3, 10 ft. from B. On the left of D, and about D, $P_1 (= 300) \cdot 20 - 500 \quad 8 = 2,000$ ft. lbs., positive bending moment at D. Or, about D, on the right side of the section; $P_2 (= 200) \cdot 10 = 2,000$ ft. lbs., positive bending moment at D. Usually compute the simpler one.

Section at A, Fig. 4, $W \quad C A = -750 \quad 5 = -3,750$ ft. lbs. negative bending moment at A, tending to make the beam convex on the upper side. At D, 10 ft. from B, $M = -P_2 \quad 10 = -187\frac{1}{2} \cdot 10 = -1,875$ ft. lbs., negative because P_2 is negative. At A, taking moments on the right of and about A, $M = -187\frac{1}{2} \quad 20 = -3,750$ ft. lbs., as first obtained. This beam has negative bending moments at all points.

In Fig. 5, M at D is $-150 \cdot 10 = -1,500$ ft. lbs. It is evident that the bending moments at all points between C and A can

be found without knowing the reactions. If this beam is built into a wall, the points of application of P_1 and P_2 are uncertain, as the pressures at A and B are distributed over more or less of the distance that the beam is embedded. The max. M is at A, and is $-150 \cdot 20 = -3,000$ ft. lbs. It is evident that the longer A B is, the smaller the reactions are, and hence the greater the security.

In Fig. 6, the bending moment at C will be P_1 A C — weight on D C . ½ D C $= 1,920 \cdot 16 - 200 \cdot 8 \quad 4 = 24,320$ ft. lbs. At E, $M = 1,280 \cdot 16 = 20,480$ ft. lbs.

In Fig. 7, the bending moments at the several points of application of the weights, taking moments of all the external forces on the left of each section about the section, will be—

At C, $M = -100 \cdot 0 = 0$.

At A, $M = -100 \cdot 2 = -200$ ft. lbs.

At D, $M = -100 \cdot 5 + (665\frac{5}{8} - 200) \cdot 3 = 896\frac{7}{8}$ ft. lbs.

At E, $M = -100 \cdot 7 + 465\frac{5}{8} \cdot 5 - 150 \cdot 2 = 1,328\frac{1}{8}$ ft. lbs.

At F, $M = -100 \cdot 10 + 465\frac{5}{8} \cdot 8 - 150 \cdot 5 - 300 \cdot 3 = 1,075$ ft. lbs.

And, at B, M will be zero. M max. occurs at E.

Do not assume that the maximum bending moment will be found at the point of application of the resultant of the load. The method for finding the point or points of maximum bending moment will be shown later.

The moments on the right portion of the beam may be more easily found by taking moments on the right side of any section. Thus at F. $M = (P_2 - 80) \cdot 8 = (214\frac{3}{8} - 80) \quad 8 = 1,075$ ft. lbs. Find the bending moment at the middle of E F. $1,201\frac{9}{16}$ ft. lbs.

In Fig. 8, the bending moment at section C of the piece A E may be found by considering the portion above C. As the unit pressure at C is $6 \times 62\frac{1}{2}$ lbs. $= 375$ lbs. per sq. ft., M at C $= (P_2 = 1,265.6) \cdot 3 - (375 \times 6 \div 2) \cdot 6 \div 3 = 1,546.8$ ft. lbs. At the section B, $M = -(3 \times 62\frac{1}{2} \times 3 \div 2) \times 1 = -281\frac{1}{4}$ ft. lbs.

In Fig. 9, as $P_1 = 400$ lbs., $P_2 = 600$ lbs., vertical forces at D and E are each 500 lbs.; M at D $= 1,600$ ft. lbs.; M at E $= 1,200$ ft. lbs.

In Fig 10, as $P_1 = -200$ lbs. $= -P_2$, and the horizontal forces at D and E are $\pm$ 500 lbs.; M at D $= -800$ ft. lbs.; M at E $= +400$ ft. lbs. The beam will be concave on the left side at D and convex at E. The curvature must change between D and E, where $M = 0$. Let this point be distant x from B. Then $200 \cdot x - 500(x - 2) = 0 \therefore x = 3\frac{1}{3}$ ft.

The curved piece A B, Fig. 11, with equal and opposite forces applied in the line connecting its ends, will experience

a bending moment, at any point D, equal to P . C D, this ordinate being perpendicular to the chord.

63. **Shearing Forces.**—In Fig. 3, of the 500 lbs. at C, 300 lbs. goes to A and 200 lbs. to B. Any vertical section between A and C must therefore have 300 lbs. acting vertically in it. On the left of such a section there will be 300 lbs. from P_1 acting upwards, and on the right of the same section there will be 300 lbs., coming from W, acting downwards. These two forces, acting in opposite directions on the two sides of the imaginary section, tend to cut the beam off, as would a pair of shears, and either of these two opposite forces is called the shearing force at the section, or simply the *shear*. When acting upwards on the left side of the section (and downwards on the right side), it is called *positive* shear. When the reverse is the case the shear will be negative.

Examples.—In Fig. 7, where a number of forces are applied to a beam, there must be found at any section between C and A a shear of —100 lbs.; between A and D the shear will be —100 + 665⅝ — 200 = + 365⅝ lbs.; between D and E the shear will decrease to 365⅝ — 150 = 215⅝ lbs.; on passing E the shear will change sign, being 215⅝ — 300 = —84⅜ lbs.; between F and B it will be —84⅜ — 50 = —134⅜ lbs.; and, on passing B, it becomes zero, a check on the accuracy of the several calculations.

In Fig. 8, the shear just above the support B = 3 × 62½ × 3 ÷ 2 = 281¼ lbs.; just below the point B the shear is 281¼— 1,265.6 = —984.4 lbs.; and just above A it is 1,265.6 lbs. The signs used imply that the left side of A E corresponds to the upper side of an ordinary beam. As the shear is positive above A and negative below B, it changes sign at some intermediate point. Find that point.

In Fig. 9, the shear anywhere between A and D is +400 lbs.; at all points between D and E it is 400 — 500 = —100 lbs.; and between E and B is —600 lbs. The shear changes sign at D.

In Fig. 10, the shear on any horizontal plane of section between B and E is —200 lbs.; between E and D is —200 + 500 = + 300 lbs., and between D and A is +300 — 500 = —200 lbs. The shear changes sign at both E and D.

64. **Summary.**—To repeat:—The shearing force at any normal section of a beam may therefore be defined to be the algebraic sum of all the transverse forces on one side of the section. When this sum or resulting force acts upward on

the left of the section, call it positive; when downward, negative.

The bending moment at any right or normal section of a beam may be stated to be the algebraic sum of the *moments* of all the transverse forces on one side of the section, taken about any point in the section as axis. When this sum or resulting moment is right-handed or clock-wise on the left of and about the section, call it positive. A positive moment tends to make the beam concave on what is usually the upper side.

By a proposition in mechanics, any force which acts at a given distance from a given point is equivalent to the same force at the point and a moment made up of the force and the perpendicular from the point to the line of action of the force. Then, in Fig. 7, if a section plane is passed anywhere, as between D and E, the resultant force on the left, which is the algebraic sum of the given forces on the left of the section, is the *shear* at the section; and this resultant, multiplied by its arm or distance from the point in D E, giving a moment which is the algebraic sum of the moments of the several forces on the left of and about the point, is the *bending moment* at the section.

It is also evident that the resulting action at any section is the sum of the several component actions; and hence that different loads may be discussed separately and their effects at any point added algebraically, if they can occur simultaneously. Thus the shears and bending moments arising from the weight of a beam itself may be determined, and to them may be *added* the shears and bending moments at the same points from other weights imposed on the beam.

The numerous examples already given show that formulas are not needed for solving problems in beams, and the student will do well to accustom himself to using the data directly. Formulas, however, will now be derived, which will sometimes be convenient for use, and from which may be deduced certain serviceable relationships.

65. **Reactions.**—If, in Fig. 7, l = distance between points of support; a = known arm of W about the point where P_2 acts, a being + when measured to the left of P_2 and

— when measured to the right, and upward forces being positive; and Σ is the sign of summation,

$$P_1\, l - \Sigma\, W\, a = 0; \text{ or } P_1 = \Sigma\, W\, a \div l.$$
$$P_2 + P_1 - \Sigma\, W = 0; \text{ or } P_2 = \Sigma\, W - P_1.$$

The same formulas will give the reactions for a beam built in at one end only, if the distance between P_1 and P_2 is known. The two reactions will then have opposite signs.

For a distributed load, weight w per unit of length of the beam, $P_1\, l = \int wda\ .\ a$, between the limits over which ω extends.

The unit load w may vary, as in Fig. 8, in which case it must be so used in the formula; or it may be constant per unit of length, as in Fig. 6. In the latter case

$$P_1 = \left.\frac{wa^2}{2l}\right] \text{ between the given limits of } a.$$

If w is constant, and covers l,

$$P_1 = \left.\frac{wa^2}{2l}\right]_0^l = \frac{w\,l}{2},$$

as is easily seen trom consideration of symmetry.

66. Shear.—The shear F, positive when acting upwards, at any section distant x' from the left end of the beam, being the sum of the transverse or perpendicular forces on the left of the section, is, for a beam fixed at the right end only, Fig. 5,

$$-F_{x'} = \Sigma_0^{x'}\, W; \text{ or for a distributed load } -F_{x'} = \int_0^{x'} wdx,$$

the integration not extending beyond the length covered by the load.

If the distance x' includes a point of support, as in the ordinary cases of beams supported at two points, Figs, 3 and 6,

$$F_{x'} = P_1 - \Sigma_0^{x'}\, W; \text{ or } F_{x'} = P_1 - \int_0^{x'} wdx,$$

for distributed loads. But, for sections to the left of P_1, the first term disappears, reducing F to the corresponding expressions above.

The shearing force in solid beams is not of much significance, on account of the amount of material which usually resists it; but in girders and trusses its determination is often necessary. It will also be required for locating the points of maximum bending moment, when simple inspection does not show them, § 68.

67. Bending Moment.—The bending moment M, positive when right handed on the left of any section, tending to make the beam concave on its upper side, will be, at a section distant x' from the left end, the sum of the moments of the forces on the left of the section, taken about a point in the section. For a beam fixed at the right end only, Fig. 5,

$$-M_{x'} = \Sigma_0^{x'}\ Wx;\ \text{or}\ -M_{x'} = \int_0^{x'} wxdx$$

for a distributed load, x being the distance of W or wdx from the section, and the integration not extending beyond the length covered by the load.

Similarly, for beams supported at two points, as usually understood, Figs. 3 and 7,

$$M_{x'} = P_1\,x - \Sigma_0^{x'}\ Wx;\ \text{or}\ M_{x'} = P_1\,x - \int_0^{x'} wxdx,$$

for distributed loads, x being, in all cases, the arm of P_1, W or wdx about the section in question, and the integration covering the loaded portion only. The unit load w may be constant or variable.

If P_1 is not to the left of the assumed section, drop the term $P_1\,x$, and obtain the preceding equations.

68. Points of Maximum Bending Moment Occur Where the Shear Changes Sign.—It will now be seen by comparison, that F is always the first derivative of M, that is $F_x = \dfrac{dM_x}{dx}$. Hence, according to the rule for determining maxima and minima, the bending moment is always a maximum (or minimum) at the place where the shear is zero or changes in sign. This criterion is easily applied to locate the points of M maximum. Pass along the beam from the left (or right) until as much load is on the left (or right) of the section as will neutralize P_1 (or P_2), and the point of M max. is

found. Its value can then be computed. If the weight at a certain point is more than enough to reduce F to zero, F changes sign in passing that point, and hence M max. occurs there.

For a beam fixed at one end only, F changes sign in passing P_1, and hence M max. is found at the wall.

Examples.—M max. occurs in Fig. 3, at C; in Fig. 4, at A; in Fig. 6, at 17.6 ft. from A; in Fig. 7, at A, and again at E; in Fig. 8, at B, and again at a distance x from E such that $62\frac{1}{2} x \cdot \frac{1}{2} x = 1265\frac{5}{8}$ $\therefore x = \sqrt{40.5} = 6.36$ ft.; in Fig. 9, at D; and in Fig. 10, at D and again at E. The bending moments which may not have been found at some of these points can now be computed.

The reader who is familiar with graphics can draw the equilibrium or bending moment polygons or curves, and the shear diagrams, and notice the same relation in them.*

The *unit load* may also be considered as the derivative of the shear; F therefore has maximum (or minimum) values where the external forces change in sign.

The origin of coördinates may be arbitrarily taken at any point in the length of the beam, and general expressions may be written. If $-w$ is the unit load, either constant or variable,

$$F_{x'} = -\int_0^{x'} w dx = F_0 - wx', \text{ if } w \text{ is constant;}$$

$$M_{x'} = -\int_0^{x'}\int_0^{x} w dx^2 = M_0 + F_0 x' - \frac{w\, x'^2}{2}, \text{ if } w \text{ is constant;}$$

where F_0 and M_0 are the constants of integration, the values of F and M at the origin, and the integration applies to the loaded portion.

A general expression for the bending moment at any point of any beam will therefore have the form $M_x = A + Bx + Cx^2$; where A is the bending moment at the origin; Bx is the sum of the moments of all the single forces, including F_0, from the origin to the point in question; and Cx^2 is the sum of the similar moments of the distributed forces about the same point. This equation is useful in more complicated cases than those at present under consideration.

69. Working Formulas.—There are a number of common cases, for which the values of F and M may be derived for convenient reference.

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*See Greene's Graphics, Part II., Bridge Trusses.

I. Beam fixed at one end only, and projecting a length l; weight W at outer end. Fig. 12. Distance x measured from wall.

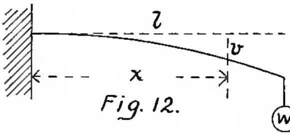

Fig. 12.

$F_x = W$, and is constant.

$M_x = -W(l - x)$, which increases with the distance from the free end and has a maximum for $x = 0$; or M max. $= -Wl$.

II. Same beam carrying a uniformly distributed load of wl. Origin at fixed end. Fig. 13.

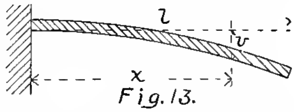

Fig. 13.

$F_x = w(l - x)$; F max. $= wl$, at wall.

$M_x = -\frac{1}{2}w(l - x)^2$; M max. $= -\frac{1}{2}wl^2 = -\frac{1}{2}(wl)l$ at wall. In these two cases the *load* and the *shear* each change sign at the wall by the addition of the upward reaction; hence F max. and M max. occur at the wall.

III. Same beam, having both a uniform load and a single weight at the outer end. Add I. and II.

F max. $= W + wl$. M max. $= -(W + \frac{1}{2}wl)l$.

IV. Beam of span l, supported at ends, carrying a single weight W in middle. Fig. 14. Origin at left point of support.

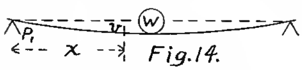

Fig. 14.

By symmetry, $P_1 = \frac{1}{2}W$.

$F_x = \frac{1}{2}W$ on left, and $= \frac{1}{2}W - W = -\frac{1}{2}W$ on right of middle.

$M_x = P_1x = \frac{1}{2}Wx$ on left, and $= \frac{1}{2}Wx - W(x - \frac{1}{2}l) = \frac{1}{2}W(l - x)$ on right. As F changes in sign at distance $\frac{1}{2}l$, M max. $= \frac{1}{4}Wl$, at middle.

V. Beam of span l, supported at ends, carrying a single weight W at known distance a from left end. Origin at left end. Fig. 15.

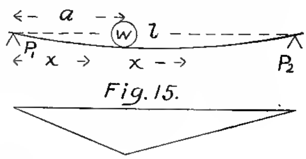

Fig. 15.

$$P_1 = W\frac{l - a}{l};\ P_2 = W - P_1 = \frac{Wa}{l}.$$

$$F_x = W\frac{l - a}{l} \text{ on the left of W; and } P_1 - W = -\frac{Wa}{l} \text{ on right.}$$

$$M_x = W\frac{l - a}{l}x \text{ on left of W; and } W\left(\frac{l - a}{l}x - (x - a)\right) = \frac{Wa}{l}(l - x) \text{ on the right of W, } = P_2(l - x).$$

M max. (where F changes sign) $= W\frac{a(l-a)}{l}$, at W.

This value may be memorized as the weight into the product of the two segments divided by the span.

VI. Beam of span l, supported at ends, and uniformly loaded with wl. Fig. 16. Origin at left.

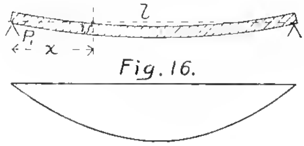

Fig. 16.

By symmetry, $P_1 = P_2 = \frac{1}{2}wl$.
$F_x = \frac{1}{2}wl - wx = w(\frac{1}{2}l - x)$.
$F = 0$, when $x = \frac{1}{2}l$. F max. $= \pm\frac{1}{2}wl$.
$M_x = \frac{1}{2}wlx - \frac{1}{2}wx^2 = \frac{1}{2}wx(l-x)$.
M_x varies as the product of the two segments.
M max. (when $x = \frac{1}{2}l$) $= \frac{1}{8}wl^2 = \frac{1}{8}(wl)l$.

VII. Case VI. can be added to IV. or V., for a uniformly loaded beam, carrying in addition a single weight. For the combination V. and VI., M max. will be found between W and the mid-span, at a distance from the middle of $Wa \div wl$. It is often easier to calculate M max. directly than to use a formula.

VIII. Beam of span l, supported at ends, and carrying a single *moving* load W Origin at left. If, at any instant, W is distant x from the left end, the shear at a point distant a from the left will be $W\frac{(l-x)}{l}$ when x is greater than a, and $-\frac{Wx}{l}$ when x is less than a. Each of these values is a maximum for $x = a$, or the positive and negative shears at any point are maximum when the weight reaches that point, and are then $\frac{W(l-a)}{l}$ and $-\frac{Wa}{l}$.

In the same way, W at x from the left will cause a moment at a point distant a from the left of $\frac{W(l-x)}{l}a$, when x is greater than a, and of $\frac{Wx}{l}(l-a)$ when x is less than a. Hence M max. at $a = \frac{Wa(l-a)}{l}$, when the weight rests on that point. The values of M max. at successive points vary as the product of the two segments, or as the

ordinates to a parabola with a vertical axis and vertex at mid-span, with middle value of $\frac{1}{4}$ Wl.

Example.—A 2,000 lb. wheel rolls across a 16 ft. span. M max. at middle = ¼ . 2,000 . 16 = 8,000 ft. lbs. At 2 ft. each way from middle M max. = 125 . 6 . 10 = 7,500 ft. lbs.; at 4 ft. = 125 . 4 . 12 = 6,000 ft. lbs., and at 6 ft. = 125 . 2 . 14 = 3,500 ft. lbs.

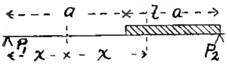
Fig. 17

IX. Beam of span l, supported at the ends and loaded uniformly over a distance $l - a$ from right end. Total load $= w(l - a)$. Origin at left. Fig. 17.

$$P_1 = \tfrac{1}{2}\,\frac{w(l-a)^2}{l};\quad P_2 = w(l-a) - P_1 = \tfrac{1}{2}\,\frac{w(l^2-a^2)}{l}.$$

$$F_x = \frac{w(l-a)^2}{2l},\ \text{when } x < a;\ \text{and } \frac{w(l-a)^2}{2l} - w(x-a),$$

when $x > a$.

$$F = 0,\ \text{when } x = a + \frac{(l-a)^2}{2l} = \frac{l^2+a^2}{2l}.$$

$$M_x = P_1\,x = \frac{w(l-a)^2}{2l}\,x,\ \text{when } x < a;\ \text{and}$$

$$= \frac{w(l-a)^2}{2l}\,x - \frac{w(x-a)^2}{2} = \frac{w(l-x)\,(lx-a^2)}{2l},$$

when $x > a$.

$$\text{M max.} = \frac{w(l^2-a^2)^2}{8l^2},\ \text{at the point } x = \frac{l^2+a^2}{2l}.$$

X. Same beam, the load advancing continuously from the right.

In this case M_x will increase as the load advances, until $a = x$, and then will continue to increase until $a = 0$, when $M_x = \frac{1}{2}wx(l - x)$ as in case VI.; and the absolute maximum M at middle $= \frac{1}{8}wl^2$

Indeed, since a single load anywhere on such a beam produces positive bending moments at all points, a complete uniform load will give maximum moments. The same is true for equally spaced, equal weights. For several independent weights, the maximum bending moment at a given point will occur when the weights are brought as near that point as possible. But when the weights differ in magnitude and are arbitrarily spaced, as in locomotive wheel weights on a bridge, a different solution is required. § 70.

F_x will increase as the load advances, until $a = x$, and will then decrease, since a is now less than x in the second value of F_x above. Hence the maximum *positive* shear at any point of this beam, from a continuous moving load, occurs when the load covers the *right* segment, of the two into which the point divides the beam. Then F_x max. $= \frac{w(l - x)^2}{2l}$; and the absolute max. shear is at the point of support, where $F_0 = \frac{1}{2}wl$.

If the figure is imagined turned end for end, it follows that the maximum *negative* shear, from a continuous moving load, occurs when the *left* segment is loaded. Then F_x max. $= -\frac{wx^2}{2l}$, with a minimum value of zero and an absolute maximum shear at the point of support of $-\frac{1}{2}wl$.

These values for F_x max., varying as the square of the distance from one or the other end, may be represented by ordinates to parabolas, drawn with the beam as tangent.

As the shear from a single weight, on a beam supported at both ends, is positive on the left and negative on the right of the weight, it will again be seen that a full load on the right segment will give maximum positive shear at a given point; and full load on the left segment, maximum negative shear.

XI. Case VI. can be added to X, for combined steady and moving uniform loads, often termed dead ($= w$) and live ($= w'$) loads, giving maximum values of

$$F_x = w(\tfrac{1}{2}\, l - x) + w'\frac{(l - x)^2}{2l}, \text{ for positive shears,}$$

which will, however, become negative near the right point of support; and

$$F_x = w(\tfrac{1}{2}l - x) - \frac{w'x^2}{2l} \text{ for negative shears,}$$

which will, however, be positive near the left abutment. At any one point these two values may be termed the maximum and minimum values of the shear for that point.

$$M_x \text{ max.} = \tfrac{1}{2}\,(w' + w)\, x\,(l - x).$$

Example.—A girder of 60 ft. span, weighing 300 lbs. per ft. and subjected to a uniform live load of 1,200 lbs. per ft. will ex-

perience a max. moment at mid-span of $\frac{1}{2}$ 1,500 · 30 · 30 = 675,000 ft. lbs. and a min. moment of 135,000 ft. lbs. At the quarter span the moments will be 506,250 ft. lbs. and 101,250 ft. lbs., or $\frac{3}{4}$ the preceding. The maximum shear at the abutments is 1,500 · 30 = 45,000 lbs. The shear at quarter span ranges from $300(30 - 15) + 1{,}200\frac{45^2}{120} = 24{,}750$ lbs., to $4{,}500 - 1{,}200\frac{15^2}{120} = 2{,}250$ lbs., and at mid-span is ± 9,000 lbs.

XII. Beam of span l, supported at both ends, and carrying two weights W, symmetrically placed at a distance a from either end. Fig. 18.

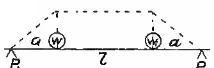

Fig. 18.

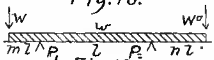

Fig. 19.

By symmetry $P_1 = P_2 = W$.

$F_x = W$, for $x < a$; $= 0$ in middle portion; $= -W$ on right.

$M_x = Wx$ on left; $= P_1x - W(x - a) = Wa$ in middle portion, and is constant, as is required where $F = 0$.

Case of cross-floorbeam for single track bridge; also trapezoidal truss, as indicated, when symmetrically loaded.

XIII. There is no difficulty in determining the reactions, shears and bending moments of cantilever beams combined with a beam supported at both ends, producing a beam supported at two points and overhanging at one or both ends. Fig. 19.

Let left cantilever $= ml$ ft., carrying W at extremity; right cantilever $= nl$ ft., carrying W′ at its end; distance between supports $= l$; total uniform load $= w(m + 1 + n)l$. With moments about P_2, $P_1 = W(m + 1) - W'n + \frac{1}{2}wl[(m + 1)^2 - n^2]$.

The shear at a distance x from the left, if $x < ml$, is $F_x = -(W + wx.)$ Upon passing P_1, $F_x = P_1 - (W + wx)$, with a probable change of sign. It may change in sign again in the length l, and again finally when P_2 is added.

Maximum negative bending moments are likely to be found at P_1 and P_2, with a maximum positive bending moment somewhere in l, if anywhere.

$$M \text{ at } P_1 = -(W + \tfrac{1}{2}wml)ml; \quad M \text{ at } P_2 = -(W' + \tfrac{1}{2}wnl)nl.$$

$$+ M \text{ max. where } x = \frac{W}{w}\,m - \frac{W'}{w}\,n + \frac{(m + 1)^2 - n^2}{2}\,l.$$

W and W′ may arise from the weight of connected beams or girders. An inspection of the preceding values will show

the effect of an increase of load on such parts. The unit load w may also change, and either m or n or both may be zero.

70. Position of Wheel Concentrations for M max. at any given Point.—Where loads have definite magnitudes and spacings, as is the case with the wheel weights of a locomotive, the position of the load, on a beam or girder supported at both ends, to give maximum bending moment at any given section may be found as follows:—

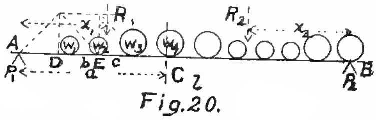

Fig. 20.

Let the given section be C, at a distance a from the left abutment of a beam A B, of span l, Fig. 20.

Let R_1 be the resultant of all loads on the left of C, and acting at a distance x_1 from the left abutment; let R_2 be the resultant of all loads to the right of C, and acting at a distance x_2 from the right abutment. The reaction P_1 at A, due to R_2 alone, will be $R_2 x_2 \div l$, and its moment about C will be $R_2 ax_2 \div l$. Similarly, P_2 at B due to R_1 alone will be $R_1 x_1 \div l$, and its moment about C will be $R_1(l - a)x_1 \div l$. Hence for the whole load we have

$$\text{M at C} = R_2 \frac{ax_2}{l} + R_1 \frac{(l-a)x_1}{l}.$$

If the entire system of loads is advanced a distance d to the left,

$$\text{M}' \text{ at C} = R_2 \frac{a(x_2 + d)}{l} + R_1 \frac{(l-a)\,(x_1 - d)}{l};$$

and M′ is greater than M at C, if $R_2 a > R_1(l - a)$, or if $\frac{R_2}{l-a} > \frac{R_1}{a}$; or by composition, if $\frac{R_1 + R_2}{l} > \frac{R_1}{a}$. If the opposite is true M is greater than M′.

But $(R_1 + R_2) \div l$ is the average load per running foot on the span, and $R_1 \div a$ is the average load per foot to the left of C. Hence the rule:—

If the average load per foot of span is greater than the average load on the left segment a, the bending moment at C will be increased by moving the system of loads to the left, and *vice versa*. A panel may be conveniently used as the unit in applying this treatment to bridge trusses.

Since, for maximum bending moment at any section, a load must be at that section, place a load W_n at the given point and compute the above *inequality*, first considering W_n as being just to the right and then just to the left of the section. If the inequality changes sign, the position with W_n at the section is one of M max. The value of M max. can then be computed as in § 62. If, however, the inequality does not change sign, move the whole system until the next W comes to the section, and test the inequality again.

It sometimes happens that two or more different positions of the load will satisfy the condition just explained, and, to determine the absolute M max., each must be worked out numerically. When there are some W's much heavier than others, M max. is likely to occur under some one of them. When other loads are brought on at the right, or pass off at the left, they must not be overlooked.

71. Position of Wheel Concentrations for Maximum Shear at any Given Point.—The shear at point C in the beam or girder, as the load comes on at the right end B, will increase until the first wheel W_1 reaches C. When that wheel passes C, the shear at that point suddenly diminishes by W_1, and then again gradually increases, until W_2 reaches C. Let R be the sum of all loads on the span when W_1 is at C, and x the distance from the centre of gravity of the loads to the right point of support B. The shear at C will be $P_1 = R\,x \div l$. If the train moves to the left a distance b, the space between W_1 and W_2, so that W_2 has just reached C, the shear at C will be $R(x + b) \div l - W_1$, plus a small quantity p, which is the increase in P_1, due to any additional loads which may have come on the span during this advance of the train. The shear at C will therefore be increased by moving up W_2, if $R\,b \div l + p > W_1$, or (as p can often be neglected), if

$$R\frac{b}{l} > W_1 \text{ or } \frac{R}{l} > \frac{W_1}{b}.$$

Hence, move up the next load when the average load per foot on the span is greater than the load on the left divided by the distance between W_1 and W_2.

Similarly, W_3 should be moved to C if $\frac{R'c}{l} > W_2$ or $\frac{R'}{l} > \frac{W_2}{c}$, R' being the sum of the loads on the span when W_2 is at C, and c the distance between W_2 and W_3.

It is not necessary to take account of p, unless the two sides of the inequality are nearly equal.

Example.—Span 60 ft., weights in units of 1,000 lbs.

	1	2	3	4	5	6	7	8	9	10	
Weights =	8	15	15	15	15	9	9	9	9	8	15
Spacing =	8′	6′	4½′	4½′	7′	5′	6′	5′	8′	8′	

To apply test for M max. at 15 ft. from left, load advancing from right. With W_2 at quarter span, load on span = 104. If W_2 is just to the right, $\frac{104}{60} > \frac{8}{15}$ or $\frac{104}{4} > \frac{8}{1}$; if W_2 is just to the left, $\frac{104}{4} > \frac{23}{1}$; therefore move up W_3 to right of quarter span. W_{10} now is on the span. $\frac{112}{4} > \frac{23}{1}$. Consider W_3 to be

just to the left; then $\frac{112}{4} < \frac{38}{1}$, or the inequality changes with W_3.
$P_1 = (8 \cdot 59 + 15 \cdot 51 + 45 \cdot 40\frac{1}{2} + 36 \cdot 21 + 8 \cdot 5) \div 60 = 64.26$.
M max. $= (64.26)15 - 8 \cdot 14 - 15 \cdot 6 = 761{,}900$ ft. lbs.

To test for F max. at same point. Put W_1 at quarter span. Load on span $= 95$. $\frac{95}{60} > \frac{8}{8}$. Move up W_2; load on span now 104. $\frac{104}{60} < \frac{15}{6}$. Inequality changes. $P_1 = (8 \quad 53 + 15 \cdot 45 + 45 \cdot 34\frac{1}{2} + 36 \cdot 15) \div 60 = 53.2$.

F max. $= P_1 - W_1 = 53.2 - 8 = 45{,}200$ lbs.

When these locomotive wheel loads are distributed to the panel joints of a bridge truss through the longitudinal stringers, which span the panel distance between floorbeams, the above rule is modified.

The load in the panel D E, being supported directly by the stringers, is by that means carried to the joints D and E. The amount thrown on D, when W_2 is at E, will be, $W_1 b \div \frac{l}{N}$, if N = number of panels in the span; and, as the reaction is $R(x + b) \div l$, the shear in the panel of the truss R is $\frac{x+b}{l} - \frac{W_1 b}{l} N$. Substitute this value in place of the previous one, and obtain

$$R\frac{b}{l} > W_1 \frac{bN}{l}, \text{ or } \frac{R}{N} > W_1.$$

Hence, move up the next load, when the whole load on the truss divided by the whole number of panels is greater than the load in the panel. The locomotive will advance farther into successive panels as it advances from the right, to give maximum positive shear. As modern panels are long, the leading wheel is not likely to pass D before the proper position is found.

72. **Maximum Bending Moment on a Beam under Moving Loads.**—When a beam or girder of uniform cross-section, such as a rolled I beam, supported at its ends, is subjected to a *system* of passing loads, such as an engine, heavy truck or trolley, it generally suffices to determine that position of the system of weights which causes the absolute maximum bending moment, the section where it is found, and its amount.

In the previous figure, let C be that section. Let R = resultant of all loads on the beam and d = its distance from B; R_1 = resultant of the loads to the left of C. The reaction at left, $P_1 = Rd \div l$; and, since the bending moment at C is to be a maximum, the shear at C must be zero, or

$$R\frac{d}{l} - R_1 = 0. \quad \therefore \quad \frac{R}{l} = \frac{R_1}{d}.$$

But the position of loads must also satisfy the condition of § 70, since there is to be maximum bending moment at C, and

$$\frac{R_1}{a} = \frac{R}{l}. \quad \therefore \frac{R_1}{a} = \frac{R_1}{d}, \text{ or } a = d.$$

The point of absolute maximum bending moment therefore is as far from one end as the centre of gravity of the whole load is from the other. This rule may be written—When the middle of the span bisects the distance between the centre of gravity of the whole load on the span and one of the wheels on either side of the centre of gravity, the desired moment is to be found under one of the two wheels.

Example.—Beam of span 24 ft. Two wheels 6 ft. apart, one carrying 2,000 lbs. and one 4,000 lbs., pass across. Centre of gravity is 2,000 . 6 ÷ 6,000 = 2 ft. from the heavier wheel. Then this wheel is to be placed 1 ft. from mid span. Reaction = 6,000 11 ÷ 24 = 2,750 lbs. M max. = 2,750 . 11 = 30,250 ft. lbs.

For beams fixed at one end and supported at the other or fixed at both ends, and for continuous girders, see Chapter VII.

73. **Compound Beam.**—When one beam upon which a load is imposed is supported by another, the bending moments and shears acting upon the latter can be found by algebraic subtraction of those resisted at corresponding sections of the former from the moments and shears of the system as a whole. That such must be the case will appear from the consideration that the bending moments and shears depend upon the weights and their positions on the span, and not upon the form of the girder or truss.

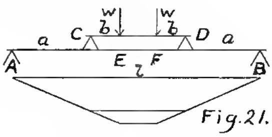

Fig. 21.

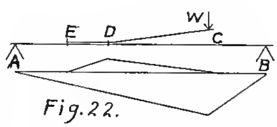

Fig. 22.

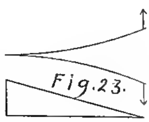
Fig. 23.

Thus in Fig. 21, by XII., §69, the bending moment between A and E, for the combination is Wx, and between E and F is W $(a + b)$; the shear from A to E is W, and is zero on E F. The shear from C to E in the upper beam is W, and the moment at E is Wb. Therefore the shear in the lower beam from C to D is zero, and the moment on that portion must be constant and equal to W $(a + b)$ — Wb = Wa. The weights on the upper beam are therefore trans-

ferred to C and D for the moments and shears on the lower beam.

Put the weights at C and D and the secondary supports at E and F, and examine the case.

In the combination of Fig. 22, by case V., § 69,

$$\text{The moment at E} = \frac{W \,.\, C\,B}{A\,B}\, A\,E\,;$$

$$\text{The combined moment at D} = \quad \text{``} \quad A\,D\,;$$

$$\text{The moment at C} = \quad \text{``} \quad A\,C.$$

The moment at D in the arm which carries W is —W . C D. Hence the moment at D in A B must be $W\left(\frac{C\,B}{A\,B}\,A\,D + C\,D\right)$, which will be greater than the moment on A B at C. This moment must also be equal to the reaction at B into B D.

In Fig. 23, the external forces being equal and directly opposed, the resultant bending moment must be everywhere zero; hence the bending moments at corresponding sections of the spring beams must be equal and of opposite signs.

Diagrams drawn below the figures show the same points clearly.

For beams of two materials: see §§ 112, 113.

74. **Total Tension Equals Total Compression.**—If a beam, loaded in any manner, and in equilibrium under the moments caused by the external forces, is cut perpendicularly across by an imaginary plane of section, while the right-handed and left-handed bending moments already shown to exist, § 62, continue to act, it is evident that the left and right segments of the beam can only be restrained from revolving about this section by the internal stresses exerted between the material particles contiguous to the section. These stresses must be of such signs, that is tensile and compressive; of such magnitude, provided the material does not give way; and so distributed over the cross-section, as to make a *resisting moment* just equal to the bending moment at the section. For the former is caused by the latter and balances it.

Since the moment arms of these stresses lie in the perpendicular plane of section, the components to be considered

now will be normal to the section. The tangential components are caused by and balance the external shear.

As the external forces which tend to bend a beam are all transverse to it, and have no horizontal components, the internal stresses of tension and compression which are caused by the bending moment must be equal and opposite, as required for a moment or couple, and hence the *total normal internal tension* on any section *must equal the total normal compression.*

When any oblique or longitudinal external forces act on a beam, there is always found that resultant normal stress on any right section which is required to give equilibrium.

75. Distribution of Internal Stress on Any Cross-Section.—It may be convenient in the beginning to consider one segment of the beam removed, and equilibrium to be assured between the external moment tending to rotate the remaining segment and the resisting moment developed in the beam at the section, as shown in Fig. 24.

If two parallel lines near together are drawn on the side of a beam, perpendicularly to its length, before it is loaded, these lines, when the beam is loaded to any reasonable amount and bent by that loading, will still be straight, as far as can be observed from most careful examination; but they will now converge to a point known as the centre of curvature for that part of the beam.

An assumption, then, that any and all right *sections* of the beam, being *plane before flexure*, *are* still *plane after* the *flexure* of this beam, is reasonable. If the right sections became warped, that warping would apparently cause a cumulative endwise movement of the particles at successive sections, especially in a beam subjected to a constant maximum bending moment over a considerable portion of its span; and such a movement and resulting distortion of the trace of the sectional plane ought therefore to become apparent to the eye. Such a warping can be perceived in shafts, other than cylindrical, subjected to a twisting couple, but cannot be found in beams.

The lines A C and B D just referred to will be found to be farther apart at the convex side of the beam, and nearer to-

gether at the concave side than they first were; hence a line G H, lying somewhere between A B and C D, is unchanged in length. If, in Fig. 24, a line parallel to A C is drawn through H, the extremity of the fibre G H which has not changed in length, K L will represent the shortening which I L has undergone in its reduction to I K, and N O will represent the lengthening which M N has experienced, in stretching to M O. The lengthening or shortening of the fibres, whose length was originally G H = ds, is directly proportional to the distance of the fibre from G H, the place of no change of length, and hence of no longitudinal or normal stress.

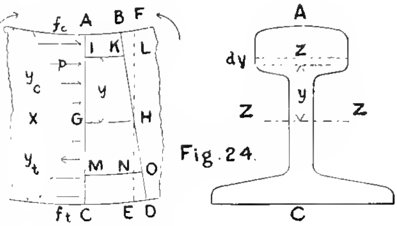

Fig. 24.

The diagram, Fig. 1, representing the elongation or shortening of a bar under increasing stresses, shows that, for stresses within the *elastic limit*, equal increments of lengthening and shortening are occasioned by equal increments of stress. If this beam has not been loaded so heavily as to produce a unit stress on any particle in excess of the elastic limit (and no working beam, one expected to last permanently, should be loaded to excess), the longitudinal unit stresses between the particles will vary as the lengthening and shortening of these fibres, that is, as the distance from the point of no stress. Hence, at any section, *the direct stress is uniformly varying*, with a maximum tension on the convex side and a maximum compression on the concave side.

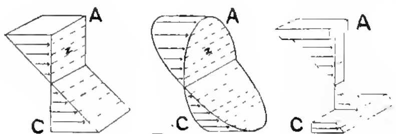

Fig. 25.

The stresses on different forms of cross-section A C are shown in Fig. 25. The total tension on the section is always equal to the total compression.

76. **Neutral Axis.**—The arrows in Figs. 24, 25 may be taken to represent the unit stress at each point of the cross-section, varying as the distance from the plane of no stress,

and constant in the direction z. To locate the point or plane of no stress or *neutral axis* for successive sections:—

Let f_c and f_t be the unit stresses of compression and tension between the particles at the extreme edge of any section, distant y_c and y_t from the point of no stress. It is plain that $f_c : f_t = y_c : y_t$ from similar triangles, and that the unit stress p at any point distant y from the point of no stress will be

$$p = \frac{f_c}{y_c}y, \text{ or } \frac{f_t}{y_t}y, \text{ or, in general } \frac{f}{y_1}y,$$

from a similar proportion.

If zdy is the area of the strip on which the unit stress p is exerted, z being the co-ordinate at right angles to x and y, the total *force* on zdy will be $pzdy = \frac{f}{y_1}yzdy = cyzdy$, where c is a constant, the unit stress at a unit distance.

As the total normal tension on the section is to equal the total compression, or their sum is to be zero, § 74, the condition may be written

$$\int_A^B pzdy = c\int_A^B yzdy = 0.$$

Therefore the sum of the moments $zdy \cdot y$ of the strips zdy about the axis of z must balance or be zero. Then the axis of z or *neutral axis must pass through the centre of gravity of* a thin plate representing *the section*, and the neutral axis of any section lies in its plane, in a direction *perpendicular to the plane of the applied external forces*. The axes of the successive cross-sections make up what is known as the *neutral plane* of the beam. Although there is no longitudinal or normal tension or compression at that line of the cross-section, it experiences shear, as will be shown later.

77. **Resisting Moment.**—The law of the variation of stress on the cross-section and the location of the neutral axis have been established. The resisting moment is caused by and is equal to the bending moment. The moments of all

the stresses about the neutral axis $Z\ Z$ is, since p has the same sign as y, and the moments conspire,

$$M = \int (+p)\, zdy\, (+y) + \int (-p)\, zdy\, (-y) = \int_A^B pyzdy.$$

As $f \div y_1$ denotes the unit stress at either extreme fibre divided by its distance from the neutral axis, and $p = \frac{f}{y_1} y$,

$$M = \frac{f}{y_1} \int_{-y_1}^{+y_1} y^2 z dy = \frac{f\,I}{y_1}; \quad \text{and } f = \frac{My_1}{I};$$

where I represents $\int y^2 zdy$ about the axis $Z\ Z$, lying in the plane of the section, through the centre of gravity of the same, and perpendicular to the plane of the external forces applied to the beam. I is termed in mechanics the *moment of inertia* of a plane area, and is usually one of the principal moments of inertia of the area. The integral will be of the fourth power, involving the breadth and the cube of the depth.

For moments of inertia of plane sections, see Chap. V

As moments of inertia for plane areas are of the *fourth* power, and can be represented by $n'bh^3$, where h is the extreme dimension parallel to y, and b to z, and as y_1 may be written $m'h$, the resisting moment can be represented, if $n' \times m' = n$, by

$$M = \frac{f\,I}{y_1} = nfbh^2,$$

n being a fraction. For a rectangular section this becomes

$$M = f\ \frac{bh^3}{12} \div \tfrac{1}{2}h = \frac{1}{6} fbh^2;$$

and for a circular section

$$M = f \cdot \frac{\pi\, d^4}{64} \div \tfrac{1}{2}d = \frac{\pi}{32} fd^3 = \frac{\pi}{4} f\, R^3.$$

Examples.—A timber beam 6 in. x 12 in., set on edge, with a safe unit stress of 800 lbs. will safely resist a bending moment amounting to $800 \cdot 6 \quad 12^2 \div 6 = 115{,}200$ in. lbs.

A round shaft, 3 in. in diameter, if $f = 12{,}000$ lbs. will have a safe resisting moment of $12{,}000 \cdot 22 \cdot 3^3 \div 7 \cdot 32 = 31{,}820$ in. lbs.

For sections other than a circle or square, either b or h is usually assumed and h or b then found. If the ratio h

$\div\ l$ is fixed by the desire to secure a certain degree of stiffness (see "Deflection of Beams," Chap. V.), the unknown quantity is b.

Example.—A wooden beam 12 ft. span, carries 3,600 lbs. uniformly distributed. $M = \frac{1}{8} Wl = \frac{1}{8} \cdot 3{,}600 \cdot 12 \cdot 12 =$ 64,800 in. lbs. If $f = 1{,}000$, $E = 1{,}400{,}000$, and the deflection v is not to exceed $\frac{1}{600}$th of the span, from $\frac{v}{l} = \frac{5fl}{48\ Ey_1}$ is obtained $\frac{1}{600} = \frac{5}{48} \cdot \frac{1{,}000 \cdot 2 \cdot 144}{1{,}400{,}000 \cdot h}$; $\therefore h = 13$ in. Then assuming $h = 14$ in., a practicable size, $64{,}800 = \frac{1{,}000}{6} b \cdot 14^2$; and $b = 2$ in.

Economy of material apparently calls for as large a value of h as possible; but the breadth b must be sufficient to give lateral stiffness to the beam, or it may fail by the buckling or sidewise flexure of the compression edge, between those points where it is stayed laterally. The effect of loading as a beam a thin board set on edge will make clear the tendency.

When the plane of the applied forces does not pass through the axis of the beam, a twisting or torsional moment is added, which will be discussed in § 93.

78. Limit of Application of $M = fI \div y_1$.—The expression for the resisting moment at any section of a beam, caused by and always equal to the external bending moment at that section, is applicable only when the maximum unit stress f does not exceed the unit stress at the elastic limit of the material. If f exceeds that limit, a uniformly varying stress over the whole section is not found, and the neutral axis may not remain at the centre of gravity. Hence, also, the substitution of breaking weights, obtained by experiments on beams which fail, in a bending moment formula which is then equated with $fI \div y_1$, results in values of f, the then so-called modulus of rupture, agreeing with neither the tensile nor the compressive strength of the material, and therefore of but limited value. This formula is correct for the purpose of design and construction; but its limitation should be kept in mind.

79. The Smaller Value of $f \div y_1$ to be Used.—Since from similar triangles $f_c \div y_c = f_t \div y_t$, it is immaterial which

ratio is used for M for a *given* cross-section. But, in designing a cross-section to resist a *given moment*, if y_t and y_c are not to be equal, another consideration has weight. A numerical example will bring out the distinction.

A beam of 24 in. span is loaded at the middle with a weight of 500 lbs. M max. will be $\frac{1}{4}Wl = 500 \cdot 6 = 3,000$ in. lbs. If the depth of the beam is 5 in., and its section is of such a form that the distance from its centre of gravity to the lower edge is 2 in., and to the upper edge is 3 in., while $I = 4$, then $3,000 = \frac{1}{2} f_t \;\; 4$ or $\frac{1}{3} f_c \cdot 4$. Hence the max. unit tension $f_t = 1,500$ lbs. per sq. in., and the max. unit compression $f_c = 2,250$ lbs. per sq. in. But if the material of the above beam must not be subjected to a unit stress greater than 2,000 lbs. per sq. in., that unit stress will be found on the compression side; for 2,000 lbs. per sq. in. on the tension side would be accompanied by 3,000 lbs. per sq. in. on the compression side; and a unit stress of 2,000 lbs. compression is only compatible in this case with $2,000 \;\; \frac{2}{3} = 1,333$ lbs. unit stress tension. The beam will safely carry only a moment of $2,000 \;\; 4 \div 3 = 2,667$ in. lbs.

Hence, when designing, with a maximum allowed value of f, and using a form of section where y_t and y_c differ, take that ratio of $f \div y_1$ which is the smaller. For a few materials, where f_c and f_t may be taken as differing in magnitude, as perhaps in cast-iron, use that ratio $f_c \div y_c$ or $f_t \div y_t$ which gives the smaller value. As the elastic limit in tension and compression for a given material is usually the same, use in computations the larger value of y_1.

80. **Inclined Beams.**—A sloping beam is to be treated like a horizontal beam, so far as resisting stress produced by that component of the load which is normal to the beam is concerned. The component of the load which acts along the beam is to be considered as producing a direct thrust along the beam if taken up at the lower end; or a direct tension, if taken up at the upper end, or as divided somewhat indeterminately, if resisted at both ends. If this longitudinal force is axial, the mean unit stress f' caused by it is to be added to the stress f'' of the same kind from bending moment at the section where this sum $f' + f''$ will be a maximum. This point can easily be found graphically. If the section of the piece is the unknown quantity, it will commonly suffice to use the value of max. M to determine an approximation to f'', and to cor-

rect the section by the resulting value of $f' + f''$ at the point where the sum is largest.

If the direct force at the end or ends is not applied axially, its moment at any section may augment or diminish the bending moment of the normal components of the load.

Cases of inclined beams, for a given load and inclination, are better solved directly than by the application of formulas.

Example. A wooden rafter, 15 ft. long, has a horizontal projection of 12 ft., and a rise of 9 ft., and it carries a uniformly distributed load of 1,500 lbs. The normal component of this load will be 1,200 lbs., the component along the roof 900 lbs. The max. bending moment, at the middle, will be $\frac{1,200 \times 15 \times 12}{8}$ $= 27,000$ in. lbs. If the safe stress is 1,000 lbs., the section to carry this moment should be $\frac{1,000\, bh^2}{6} = 27,000$, or $bh^2 = 162$. If $b = 3$, $h = 8$ in. — . If the mean thrust, at the middle of the rafter, is 1,250 lbs., the max. thrust, at the bottom end, will be 1,700 lbs., and the min. thrust, at the top end, will be 800 lbs. The section of max. fibre stress will be a very little below the middle. But, if the rafter is 3 in. $\times$ 8 in., f'' from bending moment will be $\frac{27,000 \quad 6}{3 \quad 8 \quad 8} = 844$ lbs. Also $f' = \frac{1,250}{24} = 52$ lbs. Hence $f' + f'' = 896$ lbs., a satisfactory result, if the rafter is stayed laterally by the roof covering or otherwise.

81. **Curved Beams.**—An originally curved beam, at any given cross-section made at right angles to its neutral axis, so far as the resisting stresses to bending moments are concerned, is in the same condition with an originally straight beam at a similar and equal cross-section to which the same bending moment is applied. Any definite thrust or tension at its two ends adds a moment at each right section equal to the product of the force into the perpendicular ordinate from the chord to the centre of the section, and a force, parallel to the chord, which force can be resolved into one normal to the section and a shear. Compare Fig. 11.

82. **Movement of Neutral Axis, if Yield Point is Exceeded.**—If it is assumed that cross-sections of a beam still remain plane after the yield point is passed at the extreme fibres, the stretch and shortening of the fibres at any cross-section will continue to vary with the distance from the neutral axis or plane. Suppose then that the elongation per unit of length of the outer tension fibre has attained an amount equal to O L, Fig. 1. The

unit stress on that fibre will be L N. A fibre lying half way from that edge to the neutral axis will have a unit stress K M. The total tension on the cross-section must be the area O M N L, O L now being the *distance from the neutral axis* of the beam *to the tension edge*. Since the total compression on the section must equal the total tension, an equal area O L′ M′ must be be cut off by L′ M′ and the compression curve. The neutral axis must then divide the given depth of the beam in the ratio of O L to O L′, shifting in this case towards the compression side. Had the compression curve been below the tension curve, the neutral axis would have shifted towards the convex side of the beam.

Since L N is less than L′ M′, the unit stress on the extreme fibre on the tension side is the less. Hence this displacement of the neutral axis favors the weaker side. If such action continued to the time of fracture, it would account for the fact that the application of the usual formula, $f\,I \div y_1$, to breaking moments gives a value of f which lies between the ultimate tensile and compressive strengths of the material. It must be borne in mind, however, that the compression portion of the section increases in breadth and the tension portion contracts, quite materially for ductile substances, thus adding to the complication. A soft steel bar cannot be broken by flexure as a beam at a single test.

A rectangular cross-section also tends to assume the *section* shown in Fig. 26. The compressed particles in the middle of the width can move up more readily than they can laterally, making the upper surface convex as well as wider, and the particles below at the edges, being drawn or forced in, are crowded down, making the lower surface concave as well as narrower.

Fig 26.

Hence the position of the neutral axis is uncertain, after the yield point has been passed on either face; but it is probably moved towards the stronger side.

83. **Cross-Section of Equal Strength.**—When a material will safely resist greater compression than tension, or the reverse, it is sometimes the custom to use such a form of cross-section that the centre of gravity lies nearer the weaker side. Cast-iron alone is properly used in sections of this sort. See Fig. 25, section at right. Wrought iron or steel sections are occasionally rolled or built up in a similar fashion, but the increase in width of the compression flange is then usually intended to increase its lateral stiffness.

If f_t = safe unit tensile stress, and f_c = safe unit compressive stress, the centre of gravity of the section must be found at such point that $y_t : y_c = f_t : f_c$, when the given safe stresses will occur simultaneously at the section. By com-

position, $y_t : y_c : h = f_t : f_c : f_t + f_c$, so that the centre of gravity should be distant from the bottom or top,

$$y_t = h \frac{f_t}{f_t + f_c}, \text{ or } y_c = h \frac{f_c}{f_t + f_c}.$$

Example.—If $f_t = 3{,}000$ lbs., and $f_c = 9{,}000$ lbs., $y_t = h \,.\, 3{,}000 \div 12{,}000 = \frac{1}{4} h$. If a cast-iron ⊥ section is to be used, base 10 in., thickness throughout of 1 in., and height of web h', then by moments around base, $\frac{1}{4}(h' + 1) = \dfrac{\frac{1}{2} \,.\, 10 + h'(1 + \frac{1}{2} h')}{10 + h'}$; $\therefore$ $h' = 5$ in., $y_t = 1\frac{1}{2}$ in.; $\mathrm{I} = \frac{10}{12} + \frac{125}{12} + 10 \,.\, 1 + 5 \cdot 4 = \frac{165}{4}$. $\mathrm{M} = \frac{3{,}000 \,.\, 165 \,.\, 2}{4 \cdot 3} = 82{,}500$ in. lbs., the moment that the section will carry.

84. Beam of Uniform Strength.—As has been shown in § 77, the resisting moment may be put into the form $\mathrm{M} = nfbh^2$ where n is a numerical factor depending on the form of cross-section. If, then, for a given load, bh^2 be varied at successive cross-sections to correspond with the variation of the external bending moment, the unit stress on the extreme fibre will be constant; the beam will be equally strong at all sections, except against shear; and there will be no waste of material for a given type of cross-section, provided material is not wasted in shaping.

Suppose, for example, that a beam is to be supported at its ends, to carry W at the middle, and to be rectangular in cross-section.

By § 69, IV., the bending moment at any point between one support and the middle is $\frac{1}{2}\mathrm{W}x$. Equate this value with the resisting moment. $\frac{1}{2}\mathrm{W}x = \frac{1}{6} fbh^2$. To make f constant at all cross-sections, bh^2 must vary as x from each end to the middle. If h is constant, b must vary as x, or the beam will be lozenge-shaped in plan and rectangular in elevation. If, on the other hand, b is constant, h^2 must vary as x, and the elevation will consist of two parabolas with vertices at the ends of the beam and axis horizontal, while the plan will be rectangular.

The section need not be a rectangle. If the ratio of b to h is not fixed, the treatment will be like the above; but, if that ratio is fixed, as for a circular section, or other regular figure,

$b = ch$, and h^3 must vary as the external bending moment, or, in the case above, as x. The cross-section of the cast-iron beam in the example of the previous section may be varied in accordance with these principles.

The following table gives the shape of beams of rectangular cross-section supported and loaded as stated.

BEAM.	M.	bh^2 VARIES AS	h^2 CONSTANT, b VARIES AS	b CONSTANT, h^2 VARIES AS
Fixed at one end, W at other.	$-Wx$	x	x, triangular plan, Fig. 27.	x, parabolic elevation. Fig. 28.
Fixed at one end, uniform load.	$-\frac{1}{2}wx^2$	x^2	x^2, parabolic plan, Fig. 29.	x^2, h varies as x, triangular elevation. Fig. 30.
Sup't'd both ends W at a.	$W\frac{l-a}{l}x$	x	x } triangular plan, Fig. 31.	x } parabolic elevation. Fig. 32.
	$\frac{Wa}{l}(l-x)$	$l-x$	$l-x$ }	$l-x$ }
Sup't'd both ends, uniform load.	$\frac{wx(l-x)}{2}$	$x(l-x)$	$x(l-x)$ parabolic plan. Fig. 33.	$x(l-x)$, circular or elliptical elevation. Fig. 34.

When a beam supported at both ends carries a single moving load W, passing across the beam, the bending moment at the point x, where the load is at any instant, $= Wx\ (l-x) \div l$. Such a beam will therefore fall under the last class of the above table.

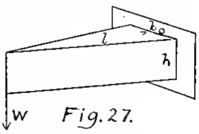
Fig. 27.

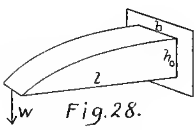
Fig. 28.

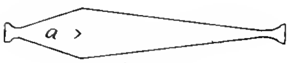
Fig. 31.

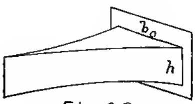
Fig. 29.

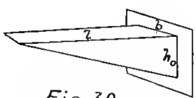
Fig. 30.


Fig. 33.

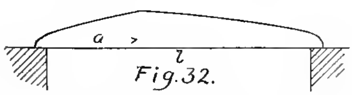
Fig. 32.

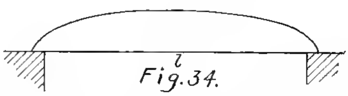
Fig. 34.

Beams which can be cast in form or built up may be made in the above outlines, if desired. Some common examples, such as brackets, girders of varying depth, working-beams, cranks, grate-bars, etc., are more or less close approximations to such forms. Enough material must also be found at any

section to resist the shear, as at the ends of beams supported at the ends, Figs. 31 to 34.

Where a plate girder is used, (see Fig. 96), with a constant depth, the cross-section of the flanges, or their thickness when their breadth is constant, will theoretically and approximately follow the fourth column of the preceding table. If the flange section is to be constant or nearly so, the depth must vary in the same way, and not as in the fifth column.

Roof and bridge trusses are beams of approximate uniform strength, for the different allowable unit stresses and for changing loads. The principles of this section have an influence on the choice of outline for such trusses, and the shapes of moment diagrams suggest truss forms.

85. Allowance for Weight of Beam.—If a beam is long and heavy, its own weight may cause a noticeable unit stress. While this weight is usually at first approximated to, or assumed, and then added to the given external load, the beam may be treated as follows:—

Design the beam or girder for the given load W′, and compute its weight B′, and breadth b' As W′ is at present all the load the beam ought to carry, the proportion exists

$$\frac{\text{Weight of beam}}{\text{Entire load}} = \frac{B'}{W'};$$

and the net *external* load the beam ought to carry will be given by the proportion,

$$\frac{\text{Entire load}}{\text{External load}} = \frac{W'}{W' - B'}.$$

As the load which a beam will carry varies with the breadth, and as it is desired to increase the net load from W′ — B′ to W′, the breadth must be increased in this ratio, or the new breadth b will give

$$\frac{b}{b'} = \frac{W'}{W' - B'}, \quad \text{or } b = b'\frac{W'}{W' - B'}.$$

As the weight, the net load and the gross load are increased in the same ratio, the weight B of the final beam, and the gross load W will be

$$B = B' \frac{W'}{W' - B'}; \quad W = \frac{W'^2}{W' - B'}.$$

W′ and B′ should have an approximately similar distribution. If different working unit stresses are allowed for B′ and W′, multiply W′ by the ratio of its unit stress to that of B′.

86. Distribution of Shearing Stress in the Section of a Beam, Pin, Etc.—It will be proved, in § 182, that, at any point in a body under stress, the unit shear on a pair of planes at right angles must be equal. Whatever can be proved true in regard to the unit shear on a *longitudinal* plane at any point in a beam must therefore be true of the unit shear on a *transverse* plane at the same point.

Fig. 35 represents a portion of a beam bent under any load. The existence of shear on planes parallel to E F is shown by the tendency of the layers to slide by one another upon flexure. Let the cross-section of the beam be constant. If the bending moment at section H, a point close to G, differs from that at G, there will be a shear on the transverse section, because the shear is the first derivative of the bending moment, § 68. The direct stress, here compression, on the face H F of the solid H F E G, will differ from that on the face G E, since the bending moments are different, and that difference will be balanced by a longitudinal horizontal force, or shear, on the plane F E, to oppose the tendency to displacement. If this force along the plane E F is divided by the area E F over which it is distributed, the longitudinal unit shear will be obtained. It follows from the first paragraph that the unit shear at the point E on the *transverse* section G A must be the same. It is also evident that the farther E F is taken from H G, the greater will be the difference between the total force on H F and that on G E, until the neutral axis is reached, and that the unit shear on the longitudinal plane E F must increase as E F approaches B, the neutral axis. The same thing is true, if the plane is supposed to lie at different distances from the edge A. Hence, at any transverse section A G, the unit shear on a longitudinal plane is most intense at the neutral axis; and therefore the unit shear on a transvêrse section A G is unequally distributed, being greatest at B, the neutral axis, and diminishing to zero at A and G.

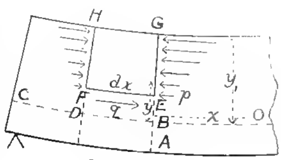

Fig. 35.

Pins and keys, and rivets which do not fit tightly in their holes, and hence are exposed to bending, have a maximum

unit shear at the centre of any cross-section, and this shear must therefore be greater than the mean value, and must determine the necessary section.

To find the mathematical expression for the variation of shear on the plane A G:—

O B D C is the trace of the neutral plane. B D = E F sensibly = dx. B E = y, B G = y_1. Breadth of beam at any point = z, at neutral axis = z_0. Normal or direct unit stress at the point E on plane A G = p. Unit shear at E = q; maximum, at B = q_0. M and F = bending moment and shearing force at section A G.

$$\text{By § 77,}\quad f = \frac{\text{M}y_1}{\text{I}} \text{ and } p = \frac{\text{M}}{\text{I}}y.$$

The total direct stress on plane G E is

$$\int_y^{y_1} pzdy = \frac{\text{M}}{\text{I}}\int_y^{y_1} yzdy. \qquad (1.)$$

The difference between M at the section through B and M at the section through D must be Fdx, since M = $\int \text{F}dx$, by § 68. The horizontal force on E F is the excess of (1.) for G E over its value for H F or $\frac{\text{F}dx}{\text{I}}\int_y^{y_1} yzdy$. Divide by the area zdx of F E over which this horizontal force acts, to find the unit shear.

$$q_y = \frac{\text{F}}{\text{I}z}\int_y^{y_1} yzdy. \quad \text{Hence } q_0 = \frac{\text{F}}{\text{I}z_0}\int_o^{y_1} yzdy.$$

Since the mean unit shear = F ÷ S, the ratio of the maximum unit shear to the mean will be found by dividing q_0 by F ÷ S.

$$\frac{\text{Max. unit shear}}{\text{Mean unit shear}} = \frac{\text{S}}{\text{I}z_0}\int_o^{y_1} yzdy = \frac{\int_o^{y_1} yzdy}{r^2 z_0},$$

where r = radius of gyration of the cross-section, and $\int yzdy$ is the moment of either the upper or lower part of the

cross-section about the trace of the neutral plane. Hence the max. unit shear will be

$$\text{Rectangle,}\quad \frac{b\int_o^{\frac{1}{2}h} ydy}{\frac{1}{12}h^2 \,.\, b} = \frac{12\,.\,h^2}{h^2\,.\,8} = \frac{3}{2},\ \text{or } 50\%\ \text{greater,}$$

$$\text{Circle,}\quad \int_o^{R} \frac{\sqrt{(R^2-y^2)}2ydy}{\frac{1}{4}R^2\,.\,2R} = \frac{2\,.\,2R^3}{R^3\,3} = \frac{4}{3},\ \text{or } 33\%\ \text{greater,}$$

Thin ring, approximately = 2 or 100 per cent. greater than the mean unit shear.

For beams of variable cross-section I will not be constant; but the preceding results are near enough the truth for practical purposes.

Example.—A 4 in. × 6 in. beam has at a certain section a shear of 2,400 lbs.; the max. unit shear on both the horizontal and vertical plane, at the middle of the depth, is $\frac{2,400}{24} \cdot \frac{3}{2} = 150$ lbs. on the sq. in.

As shearing resistance along the grain of timber is much less than the shearing resistance across the grain, wooden beams which fail by shearing fracture along the grain at or near the neutral axis, at that section where the external shear is greatest. As the unit shears on two planes at right angles through a given point are always equal, the shearing strength of timber across the grain cannot be availed of, since the piece will always shear along the grain.

Example—A cylindrical bridge pin 3 in. diam., area 7.07 sq. in., has a shear of 50,000 lbs. The max. unit shear is $\frac{50,000}{7.07} \quad \frac{4}{3}$ = 9,430 lbs. per sq. in.

If the apparent allowable unit shear is reduced one quarter, as from 10,000 lbs. to 7,500 lbs., the same circular section for a pin will be obtained in designing, as if the maximum unit shear were considered. For a rectangular section the apparent allowable shear should be reduced one-third.

87. **Variation of Unit Shear.**—The distribution of shear on three forms of cross section is indicated in Fig. 36, where the ordinates show the unit shear at corresponding points. For the rectangle the curve is a parabola, as the breadth of

the section is a constant. For the circle, the parabola ordinates are divided by a varying breadth, altering the curve as below. The curve for the I shaped section will be made of three parabolas as shown, but the unit shear on the flanges will be given in due proportion by dividing the ordinates of the dotted parabolas by the ratio of width of flange to thickness of web, giving the full curve. The unit shear for the I section

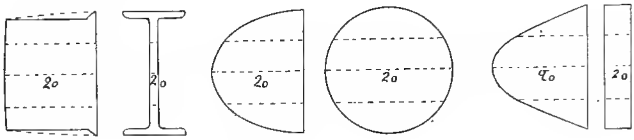

Fig. 36.

is thus shown to be practically constant over the web, and to differ but little from the unit shear found by dividing the total shear at a section by the cross-section of the web, as is usually done in practice.

Examples.—1. Three men carry a uniform timber 30 ft. long. One man holds one end of the timber; the other two support the beam on a handspike between them. Place the handspike so that each of the two shall carry 3/8 of the weight.

2. Three sections of water pipe, each 12 ft. long, are leaded end to end. In lowering them into the trench, where shall the two slings be placed so that the joints will not be strained? Neglect the extra weight of socket.

3. Wooden floor joists of 14 ft. span and spaced 12 in. from centre to centre are expected to carry a floor load of 80 lbs. per sq. ft. If $f = 900$ lbs., what is a suitable size? 2 in. × 10 in.

4. One of these joists comes at the side of an opening, 4 ft. by 6 ft., the load from the shorter joists, then 10 ft. long, being brought on this longer joist at 4 ft. from one end. How thick should this joist be? 4 in.

5. A cylindrical water tank, radius 20 ft., is supported on I beams radiating from the centre. These beams are supported at one end under the centre of the tank and also on a circular girder of 15 ft. radius. They are spaced 3 ft. apart at their outer extremities. If the load is 2,000 lbs. per sq. ft. of bottom of tank, find the max. + and — M on a beam.

+ 29,600; —68,750 ft. lbs.

6. Find b and h for the strongest rectangular beam that can be sawed from a round log of diameter d.

7. An opening 10 ft. wide, in a 16 in. brick wall, is spanned by a beam supported at its ends. The max. load will be a triangle of brick 3 ft. high at the mid-span. If the brickwork weighs 112 lbs. per c. ft., find M at the mid-span. 44,800 in. lbs.

8. In the above problem, write an expression for M at any point, if w = weight of unit volume of load, a = height of load at middle, t = thickness of wall, l = span and x = distance of point from the support.

9. A trolley weighing 2,000 lbs. runs across a beam 6 in. wide and of 20 ft. span. What will be the elevation of a beam of uniform strength, and what its depth at middle, if $f = 800$ lbs?

12 in. +

10. A round steel pin is acted upon by two forces perpendicular to its axis, a thrust of 3,000 lbs. applied at 8 in. from the fixed end of the pin, and a pull of 2,000 lbs. applied 6 in. from the fixed end and making an angle of 60° with the direction of the first force. Find the size of the pin, if $f = 8,000$ lbs.

M = 20,784 in. lbs.

11. A beam of 20 ft. span carries two wheels 6 ft. apart longitudinally, and weighing 8,000 lbs. each. When they pass across the span, where and what is the max. M? 57,800 ft. lbs.

12. A floor beam for a bridge spans the roadway a and projects under each sidewalk b. If dead load per foot is w, live load for roadway w', for sidewalk w'', write expressions for + M max. and — M max.

CHAPTER IV.

TORSION.

88. **Torsional Moment.**—If a uniform cylindrical bar is twisted by applying equal and opposite couples or moments at two points of the axis, the planes of the couples being perpendicular to that axis, the particles on one side of a cross-section tend to rotate about the axis and past the particles on the other side of the section, thus developing a shearing stress that varies with the tendency to displacement of the particles, that is, directly as the distance of each particle from the centre. The unit shear then is constant on any *ring*, and the shearing stresses thus set up at any section make up the resisting moment to the torsional moment of the applied couple. As all cross-sections are equal and the torsional moment is constant between the two points first referred to, each longitudinal fibre will take the form of a helix.

89. **Torsional Moment of a Cylinder.**—If the unit shear at the circumference of the outer circle, Fig. 37, of radius r_1 and diameter d is q_1, the value at a distance r from the centre will be, by the above statement, $q = q_1 r \div r_1$. The total shearing force on the face of an infinitesimal particle whose lever arm is r, and area $r dr d\theta$, will be $\frac{q_1}{r_1} r^2 dr d\theta$, and its moment about the centre will be $\frac{q_1}{r_1} r^3 dr d\theta$. Hence the resisting moment

$$T = \frac{q_1}{r_1}\int_0^{r_1}\int_0^{2\pi} r^3 dr d\theta = \tfrac{1}{2}\pi\, q_1 r_1^3 = \frac{\pi q_1 d^3}{16} = 0.196\ q_1 d^3 \text{ for a}$$

cylinder.

As the quantity to be integrated is $r^2 \,.\, r dr d\theta$, or is the summation of the products of the areas $r dr d\theta$ by the squares of their distances from the axis, this integral is the *moment of inertia of the section about an axis through its centre and*

perpendicular to its plane, known as the polar moment of inertia, or J. If y and z lie in the plane of the section, and x lies in the axis of the shaft, $\int mr^2 = \int my^2 + \int mz^2$, and in general $J = I_y + I_z$ for any form of cross-section; for a circle $J = 2I_z = \frac{1}{2}\pi r_1^4$.

Hence the resisting moment against torsion may be written $T = q_1 J \div r_1$, which resembles in form the resisting moment against flexure, but differs in using the polar moment of inertia of the cross-section for the rectangular one, and in having q_1, max. unit shear in place of f, max. unit tension or compression.

90. **Torsional Moment of a Square Shaft.**—If a square bar is twisted and the shear is assumed to vary on the

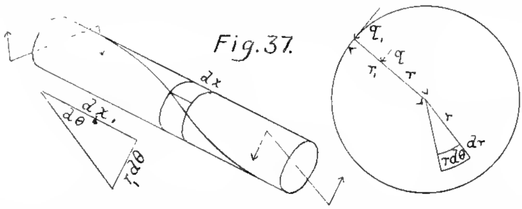

Fig. 37.

cross-section with the distance of the particles from the centre, $J = \frac{1}{6}h^4$, $r_1 = h\sqrt{\frac{1}{2}}$ and

$$T = \frac{q_1}{h\sqrt{\frac{1}{2}}}\cdot\frac{h^4}{6} = 0.236\ q_1h^3.$$

This assumption is not correct. The unit shear is actually the greatest at the middle of each side. For rectangular sections the preceding treatment would be seriously in error, but for a square section the error is not important. The last coefficient should be about 0.208. For shafts the cylindrical form is now almost universal. See § 92.

The exact investigation for sections other than circles is very involved. See "Theory of the Elasticity of Solid Bodies," Clebsch: translated from the German into French, with Notes by de St. Venant and Flamant; and Report of Chief of Engineers, U. S. A., for 1895, p. 3041, Part IV.

Example.—A round shaft, $2\frac{1}{2}$ in. diam. carries a pulley of 30 in. diam.; the difference of tension on the two parts of the belt is 1,000 lbs. Then $T = 1{,}000 \cdot 15$; $q_1 = \dfrac{16 \cdot 15{,}000 \cdot 7 \cdot 2^3}{22 \quad 5^3}$ $= 4{,}887$ lbs. per sq. in., if the torsional moment is entirely carried by the section of the shaft on one side of the pulley.

91. The Twist of a Cylindrical Shaft.—If $d\theta$ is the small angle at the centre that the radius revolves in passing longitudinally a distance dx, the distortion is $r_1 d\theta \div dx$, and, as the pitch of the helix is regular, $d\theta \div dx = \theta \div x$. If q_1 is the unit shear at the point whose radius is r_1, and the modulus of elasticity for shear, by definition, § 10, $C = q_1 \div$ distortion,

$$\frac{r_1 d\theta}{dx} = \frac{q_1}{C}; \text{ or } \frac{d\theta}{dx} = \frac{\theta}{x} = \frac{q_1}{Cr_1}. \quad \therefore \theta = \frac{q_1 x}{Cr_1} = \frac{2q_1 l}{Cd}$$

where l is the distance between the points of application of the two couples. As

$$T = \frac{\pi}{16} q_1 d^3 \text{ for a round shaft, } q_1 = \frac{16\,T}{\pi d^3} \text{ and}$$

$$\theta = \frac{32\,Tl}{\pi C d^4} = \frac{10\,Tl}{Cd^4}, \text{ nearly.}$$

If, for a square shaft, $T = 0.208\, q_1 h^3$,

$$\theta = \frac{Tl}{0.104\, Ch^4} = \frac{9.6\, Tl}{Ch^4}, \text{ nearly.}$$

Example.—If in the preceding example the length of shaft subject to this twisting moment is 30 ft. = 360 in., and C = 9,000,000, $\theta = \dfrac{10 \,.\, 15{,}000 \,.\, 360 \,.\, 2^4}{9{,}000{,}000 \,.\, 5^4} = 0.154$. To reduce this angle to degrees multiply by $\dfrac{180}{\pi}$ or 57.3, giving $\theta = 8^\circ\, 50'$, nearly.

92. St. Venant's Equations for Torsion.—When a torsional moment is applied to a body whose section is not a circle, the following equations have been given by M. St. Venant for the max. unit stress produced, which is found at *points* of the boundary *nearest the centre*:—

$q_1 = \frac{3}{4} \dfrac{T}{I} \dfrac{b}{2}$ for a rectangle whose shorter side is b and whose moment of inertia is taken through the centre of gravity about an axis parallel to the longer side.

$q_1 = \frac{1}{2} \dfrac{T}{I} b$ for an ellipse whose least semi-diameter is b and whose moment of inertia is taken about the greatest diameter.

The torsional angle for unity of length

$$\theta = \text{about } 40 \frac{T\,J}{C\,S^4}, \text{ for rectangle, and } = 4\,\pi^2 \frac{T\,J}{C\,S^4},$$

for ellipse; where J = polar moment of inertia about axis through the centre of gravity, and S = cross-section.

93. Effect of Twisting on a Beam.—Combination of bending moment M and torsional moment T. Fig. 38.

The normal unit stress on the cross-section of the extreme fibre, from the bending moment on the beam or shaft, is f, tension on one edge, compression on the opposite edge. The unit shear on the cross-section of the same extreme fibre is q_1. Refer to § 197, and remember that, at that point, there is an equal unit shear on the plane at right angles to the cross-section; then there are given f and q_1 on the cross-sectional plane, and q_1 on the plane at right-angles, to find the direction and magnitude of the new principal unit stress.

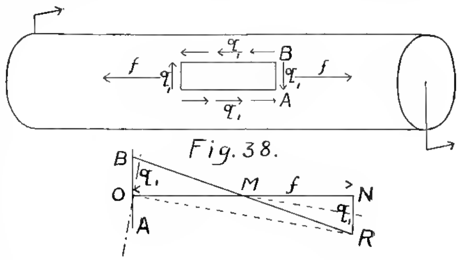

Fig. 38.

A B = plane of cross-section; O N $= f$, N R $= q_1$, laid off in succession. Then O R = resultant unit stress on A B. O B $= q_1$ on second plane, revolved 90° to make the two planes and normals coincide. Draw B R connecting the extremities of the two stresses. As its middle point falls on the middle point M of O N,

$$p_1 = \text{O M} + \text{M R}. \quad \text{M R}^2 = \text{M N}^2 + \text{N R}^2. \quad \therefore \quad \S\ 197,$$

$$p_1 = \tfrac{1}{2}f + \sqrt{(\tfrac{1}{4}f^2 + q_1^2)}. \qquad (1.)$$

$$p_2 = \text{O M} - \text{M R},$$

which will be opposite in kind to p_1, since M R > O M. The direction of p_1 is parallel to the line bisecting the angle N M R.

Since, by § 77, $M = f\ I_z \div y_1$, where I_z = rectangular moment of inertia of cross-section, and, by § 89, $T = q_1\ J \div y_1$, where J = polar moment of inertia of the same section, and since $J = 2\ I_z$ for a circle or a square, § 89, (1.) may be multipled by $I_z \div y_1$, and transformed to

$$M_1 = \tfrac{1}{2}M + \sqrt{(\tfrac{1}{4}M^2 + \tfrac{1}{4}T^2)} = \tfrac{1}{2}(M + \sqrt{(M^2 + T^2)}\,) \qquad (2.)$$

where M = original bending moment at the section, T = original torsional moment, and M_1 = equivalent resultant bending moment for which the beam or shaft should be designed, so that the unit stress shall not exceed f when both M and T occur at the given section.

Some authors multiply (1.) by $J \div y_1$, and produce

$$T_1 = M + \sqrt{(M^2 + T^2)}$$

as an equivalent torsional moment, to be used when T is much larger than M.

As the section on which the new principal unit stress acts is perpendicular to a line bisecting the angle N M R, shown by the broken line through O, this section is not quite the same as the original right section, and hence a small inaccuracy is involved in (2.).

Examples.—1. If the pulley of the previous example weighs 500 lbs. and is 18 in. from the hanger, on a free end of the shaft, and the unbalanced belt pull of 1,000 lbs. is horizontal, the resultant force will be 1,118 lbs., and a bending moment of 20,124 in. lbs. will be felt at the hanger. Then $M_1 = \frac{1}{2}(20,124 + \sqrt{(20,124^2 + 15,000^2)}) = \frac{1}{2}(20,124 + 25,100) = 22,612$ in. lbs., which will cause a fibre stress of $\frac{22,612 \cdot 7 \cdot 32 \cdot 2^3}{22 \cdot 5^3} = 14,735$ lbs.

2. The wooden roller of a windlass is 4 ft. between bearings. What should be its diameter to safely lift 4,000 lbs. with a 2 in. rope and a crank at each end, both cranks being used and f being 800 lbs.? $8\frac{1}{2}$ in.

3. Design a shaft to transmit 500 horse power at 80 revolutions per min., if $q = 9,000$ lbs. $d = 6$ in.

4. How large a shaft will be required to resist a torsional moment of 1,600 ft. lbs. if $q = 7,500$ lbs.? If the shaft is 75 ft. long and $C = 11,200,000$, what will be the angle of torsion?

1 in.; 45°.

CHAPTER V.

MOMENTS OF INERTIA.

94. **Moments of Inertia.**—Values of I for the more common forms of cross-section S. Also of $r^2 = I \div S$.

I. Rectangle, height h, base b. Fig. 39. Axis through the centre of gravity and parallel to b.

$$I_z = \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} y^2 z dy = b \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} y^2 dy = \left[\frac{1}{3} b y^3\right]_{-\frac{1}{2}h}^{+\frac{1}{2}h}$$

$$= \frac{bh^3}{24} + \frac{bh^3}{24} = \frac{bh^3}{12}. \qquad I_y = \frac{b^3 h}{12}.$$

$$r^2 = \frac{bh^3}{12} \div bh = \frac{h^2}{12}.$$

For an axis through the centre of gravity and perpendicular to the plane,

$$J = I_y + I_z = \frac{bh}{12}(b^2 + h^2), \text{ and } r^2 = \frac{1}{12}(b^2 + h^2).$$

II. Triangle, height h, base b. Fig. 40. Axis as above and parallel to b.

$$h \,.\, b = \frac{2}{3} h - y : z; \quad z = \frac{b}{h}\left(\frac{2}{3} h - y\right).$$

$$I_z = \int_{-\frac{1}{3}h}^{-\frac{2}{3}h} y^2 z dy = \frac{b}{h}\int_{-\frac{1}{3}h}^{-\frac{2}{3}h} \left(\frac{2}{3} h - y\right) y^2 dy = \frac{b}{h}\left[\frac{2}{9} h\, y^3\right.$$

$$\left. - \frac{1}{4} y^4\right]_{-\frac{1}{3}h}^{\frac{2}{3}h} = \frac{b}{h}\left(\frac{16}{243} - \frac{16}{324} + \frac{2}{243} + \frac{1}{324}\right) h^4 = \frac{bh^3}{36}.$$

$$r^2 = \frac{bh^3}{36} \div \frac{bh}{2} = \frac{h^2}{18}.$$

III. Isosceles triangle, about axis of symmetry. Fig. 41. Height along axis h, base b.

$$h \;.\; \tfrac{1}{2}b = z : \tfrac{1}{2}b - y; \quad z = h\left(1 - \frac{2y}{b}\right).$$

$$I_z = 2\int_0^{\frac{1}{2}b} h\left(1 - \frac{2y}{b}\right) y^2 dy = 2h\left[\frac{y^3}{3} - \frac{y^4}{2b}\right]_0^{\frac{1}{2}b}$$

$$= 2h\left(\frac{b^3}{24} - \frac{b^3}{32}\right) = \frac{hb^3}{48}. \qquad r^2 = \frac{b^2}{24}.$$

The sum of II and III will be the polar moment J, about an axis through the centre of gravity and perpendicular to the plane.

$$J = \frac{bh}{12}\left(\frac{h^2}{3} + \frac{b^2}{4}\right); \quad r^2 = \frac{1}{6}\left(\frac{h^2}{3} + \frac{b^2}{4}\right).$$

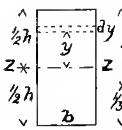
Fig. 39.

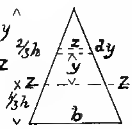
Fig. 40.

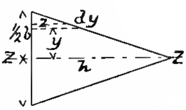
Fig. 41.

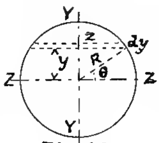
Fig. 42.

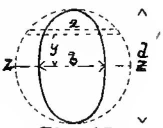
Fig. 43.

IV. Circle, radius R, diameter d. Fig. 42. If θ = angle between the axis of z and a radius drawn to the extremity of any element parallel to z,

$$y = R\sin\theta; \quad \tfrac{1}{2}z = R\cos\theta; \quad dy = R\cos\theta\, d\theta.$$

$$I_z = 4R^4\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\, d\theta = -2R^4 \;.\; \tfrac{1}{8}\left[\tfrac{1}{4}\sin 4\theta - \theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi R^4}{4} = \frac{\pi d^4}{64}. \qquad r^2 = \tfrac{1}{4} \;.\; R^4 \div \pi R^2 = \tfrac{1}{4}R^2 = \frac{1}{16}d^2.$$

The polar moment of inertia J, may be easily written if r' = variable radius,

$$J = \int_0^R r'^2 \;.\; 2\pi r'\, dr' = \frac{\pi R^4}{2}. \qquad r^2 = \tfrac{1}{2}R^2$$

Since $I_z + I_y = J$, and $I_y = I_z$ by symmetry, $I_z = \tfrac{1}{4}\pi R^4$ as before.

V. Ellipse. Diameters d and b. Fig. 43.

As the value of z in the ellipse is to that of z in the circle, as the respective horizontal diameters, or as b to d, and as the moment of the strip zdy varies as the breadth alone, the ellipse having horizontal diameter b, height d, gives

$$I_z = \frac{\pi d^4}{64} \cdot \frac{b}{d} = \frac{\pi b d^3}{64}; \quad r^2 = I_z \div \frac{\pi b d}{4} = \frac{d^2}{16}.$$

$$I_y\text{, by analogy} = \frac{\pi d b^3}{64}. \quad J = \frac{\pi b d}{64}(d^2 + b^2); \quad r^2 = \frac{1}{16}(d^2 + b^2).$$

VI. The moment of inertia of a hollow section, when the areas bounded respectively by the exterior and interior perimeters have a common axis through their centres of gravity, can be found by subtracting I for the latter from I for the former. Thus,

Hollow rectangle, interior dimensions b' and h', exterior b and h; $I_z = \frac{1}{12}(bh^3 - b'h'^3)$.

Hollow circle, interior radius R′, exterior radius R; $I_z = \frac{1}{4}\pi(R^4 - R'^4)$.

The moment of inertia of a hollow ring of outside diameter d and inside diameter d', the ratio of d' to d being n, may be written

$$I_z = \frac{1}{64}\pi(d^4 - d'^4) = \frac{1}{64}\pi(1 - n^4)\, d^4$$

As the cross-section $S = \frac{1}{4}\pi(d^2 - d'^2) = \frac{1}{4}\pi(1 - n^2)\, d^2$;
$n^2 = 1 - \frac{4\,S}{\pi d^2}$, and $I_z = \frac{S}{8}\left(d^2 - \frac{2\,S}{\pi}\right) = \frac{S}{8}\left(d^2 - \frac{7S}{11}\right)$.

95. Moment of Inertia About a Parallel Axis.—To find the moment of inertia I′ of a plane area about an axis z parallel to the axis z_0 through the centre of gravity and distant c from it.

By definition $I' = \int (y + c)^2 z dy = \int y^2 z dy + 2c \int y z dy + c^2 \int z dy$. The first term of the second member is I_z, the moment of inertia about the axis through the centre of gravity; the second term has for its integral the moment of the area about its centre of gravity, which moment is zero;

and the integral in the third term is the given area S. Hence, $I' = I_z + c^2 S = (r^2 + c^2) S = r'^2 S$.

Example.—I_z for rectangle, axis parallel to b, is $\frac{1}{12}bh^3$. I' about base $= \frac{1}{12}bh^3 + \frac{1}{4}h^2 \cdot bh = \frac{1}{3}bh^3$; and $r'^2 = \frac{1}{3}h^2$.

The reverse process is convenient for use.

$$I_z = I - c^2 S = (r'^2 - c^2) S = r^2 S.$$

As the value of I about an axis through the centre of gravity is the least of all I's about parallel axes, it can readily be seen whether c^2 S is to be added or subtracted.

It is frequently necessary to divide areas, such as T, I and built iron sections, and those of irregular outline, into parts whose moments of inertia are known, each about an axis through its own centre of gravity; then, to the sum of their several I's, add the sum of the products of each smaller

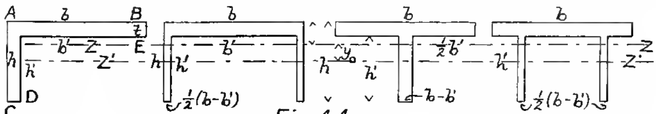

Fig. 44.

area into the square of the distance from its axis to the parallel axis through the centre of gravity of the whole. This rule is an expansion of the preceding one.

$$I_z \text{ for the whole} = \Sigma I + \Sigma c^2 S.$$

If the value of I_1 about an axis distant c_1 from the centre of gravity is known, and it is desired to find I_2 about a parallel axis distant c_2 from the centre of gravity a combination of the two formulas

$$I_1 = I_z + c_1^2 S, \text{ and } I_2 = I_z + c_2^2 S \text{ gives}$$
$$I_2 = I_1 + (c_2^2 - c_1^2) S.$$

Example.—I for a triangle about an axis through the vertex parallel to the base is easily obtained, since $z : b = y : h$.

$$\text{Therefore I about vertex} = \int_o^h \frac{b}{h} y^3 dy = \frac{bh^3}{4}$$

$$\text{Then I about base} = \frac{bh^3}{4} + \left(\frac{1}{9} - \frac{4}{9}\right) h^2 \frac{bh}{2} = \frac{bh^3}{12}$$

96. **Moments of Inertia, Continued.**—VII. L, T, Channel or TT section. Fig. 44. Area $= bh - b'h' = A - A'$. To find y_0 = distance of centre of gravity from axis

though $\frac{1}{2}h$, parallel to b. Area bh will have no moment about $Z'Z'$

$$y_0 = \frac{b'h'(\frac{1}{2}h - \frac{1}{2}h')}{bh - b'h'} = \frac{A'(h - h')}{2(A - A')} = \frac{A't}{2(A - A')} = \frac{A't}{2S}.$$

$$\text{Then } I_z = \frac{bh^3}{12} - \frac{b'h'^3}{12} + bhy_0^2 - b'h'[\tfrac{1}{2}(h - h') + y_0]^2 =$$

$$A\left(\frac{h^2}{12} + y_0^2\right) - A'\left[\frac{h'^2}{12} + \left(\frac{t}{2} + y_0\right)^2\right]$$

I about edge A B or C D of L $= I_z + (A - A')\,(\frac{1}{2}h \pm y_0)^2$. By interchange of b and h, values of I about A C and B E are obtained. These values doubled will give I_y, for the channel and T sections.

Example.—$h = 6$in., $h' = 5$in., $b = 8$in., $b' = 7$in., $S =$ 13 sq. in. $A' = 35$ in. $t = 1$ in. $y_0 = \frac{35 \cdot 1}{2 \cdot 13} = 1.35$ in. —. $I_z = 48(3 + 1.8) - 35\ (2.1 + 3.4) = 37.9$.

Formulas for such cases are of little value. In actual computations follow the general rule.

VIII. I_z, for such symmetrical sections as shown in Fig. 45, can be readily calculated by writing the value of I for the the exterior bounding rectangle and subtracting the I's for rectangles indicated by the dotted lines.

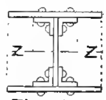

Fig. 45.

97. **Axes of Symmetry.**—The following facts have useful applications. If a plane area has two axes of symmetry not at right angles to each other, its moment of inertia is the same about all axes lying in it and passing through its centre of gravity. Examples—equilateral triangle, square, regular pentagon, hexagon, etc. I may be calculated, therefore, about that axis which gives the simplest relations.

Example.—Hexagon, side a. Axis through opposite vertices. Each half composed of one rectangle $a\sqrt{\frac{3}{4}}\ a$, and two triangles of base $\frac{1}{2}a$ and altitude $a\sqrt{\frac{3}{4}}$. I for rectangle about base $= \frac{1}{3}bh^3 = \frac{1}{3}a^4\sqrt{(\frac{3}{4})^3}$; I for one triangle about base $= \frac{1}{12}bh^3 = \frac{1}{12}\ \frac{1}{2}a^4\sqrt{(\frac{3}{4})^3}$.

I_z for hexagon $= 2(\frac{1}{4}a^4\sqrt{\frac{3}{4}} + \frac{1}{16}a^4\sqrt{\frac{3}{4}}) = \frac{5}{8}a^4\sqrt{\frac{3}{4}}$.

Area, $S = 2a^2\sqrt{\frac{3}{4}} + a^2\sqrt{\frac{3}{4}} = 3a^2\sqrt{\frac{3}{4}}$. $r^2 = \frac{5}{24}a^2$.

The polar moment of inertia, about the axis x, is equal to the sum of the two moments about y and z. As, in general, y and z may have any directions at right angles to one another, the sum of I_y and I_z must always be a constant for a given area

Two sections which have the same value for I_z do not have the same resisting moment unless y_1 is also the same in both cases.

98. Resisting Moment about an Oblique Axis.—When the plane of the external forces passes through the axis of the beam but is not parallel to either h or b, the maximum values of f or the value of M max. can be found as follows:—Fig. 46.

The section is A B E F; its centre of gravity is G; the plane of the applied forces and of flexure is N N; Y Y and Z Z are the usual rectangular axes, and the angle of axis N N with Y Y is θ.

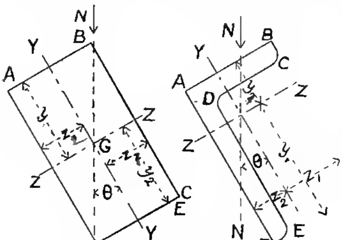

Fig. 46.

Let y_1 and y_2 denote the distances of axis Z Z from the edges A B and E F respectively and z_1 and z_2 the similar distances of axis Y Y from B C and A F

If M = bending moment of the external forces at this section, the component in the plane of y will be M cos θ, and that in the plane of z will be M sin θ. The unit-stress on the layer A B from the former will be $f' = M \cos \theta y_1 \div I_z$ and on E F, $= M \cos \theta y_2 \div I_z$. The stress on B C from the latter component moment will be $f'' = M \sin \theta z_1 \div I_y$, and on A F, $= M \sin \theta z_2 \div I_y$.

The points A, B, and F have stresses equal to the algebraic sum $f' \pm f''$ or

$$f = M\left(\frac{(y_1 \text{ or } y_2)\cos\theta}{I_z} \pm \frac{(z_1 \text{ or } z_2)\sin\theta}{I_y}\right).$$

It is plain that the corners or points B and F have the maximum unit stresses in the above figures, as the sign of the second term in the above formula for these points will be +.

$$f \text{ at B} = M\left(\frac{y_1 \cos\theta}{I_z} + \frac{z_1 \sin\theta}{I_y}\right), \text{ compression, if M is } +.$$

$$f \text{ at F} = \text{M}\left(\frac{y_2 \cos\theta}{\text{I}_z} + \frac{z_2 \sin\theta}{\text{I}_y}\right), \text{ tension, if M is } +.$$

$$f \text{ at A} = \text{M}\left(\frac{z_2 \sin\theta}{\text{I}_y} - \frac{y_1 \cos\theta}{\text{I}_z}\right).$$

If $y_1 = y_2$ or $z_1 = z_2$, the expression is simpler.

Example,—8 in. steel I beam, purlin on a roof that slopes at 30° to horizon. Purlin normal to roof, load vertical, span $12\frac{1}{2}$ ft., area carried $12\frac{1}{2} \times 8$ ft., 30 lbs. per ft. Total load = 3,000 lbs. $\text{M} = 3{,}000 \times 150 \div 8 = 56{,}250$ in. lbs. Width of flange $4\frac{1}{4}$; $\text{I}_z = 57.8$; $\text{I}_y = 4.35$.

$$f = 56{,}250\left(\frac{4 \times 0.866}{57.8} + \frac{2.12 \times 0.5}{4.35}\right) = 56{,}250\ (0.06 + 0.24)$$

= 16,875 lbs., a value which is somewhat too large, especially if the assumed load is not a liberal estimate and if wind pressure is to be added.

99. Moments of Inertia for Thin Sections.—Values of I for rolled shapes may also be approximately obtained by the following method, and, if the values given in the manufacturers' hand-books are not at hand, will prove serviceable.

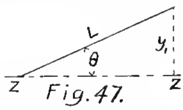

Fig. 47.

The moment of inertia of a thin strip or rod, Fig. 47, of length L and thickness t about an axis passing through one end of, and making an angle θ with it, is the same as if $\frac{1}{3}\ t\text{L}$ were concentrated at the extreme end. Let l = distance along strip to any particle.

$$\text{I} = \int_0^{\text{L}} t\,dl \,.\, l^2 \sin^2\theta = \tfrac{1}{3}\, t\text{L}^3 \sin^2\theta = \tfrac{1}{3}\, t\text{L}y_1^2,$$

or one-third the area multiplied by the square of the ordinate to the extreme end.

This expression might be derived from I for a rectangle, taken about one base.

If the rod is parallel to the axis, and at a distance y_1 from it, $\text{I} = t\text{L} \,.\, y_1^2$, since all particles are equidistant from the axis. By the application of these two formulas the following values are obtained.

I. Hollow rectangle, sides b and h; b parallel to axis.
 I for two sides, b, $= 2bt(\frac{1}{2}h)^2 = \frac{1}{2}\,bth^2$.
 I for 4 pieces, $\frac{1}{2}h$, $= 4 \,.\, \frac{1}{3}\ \ \frac{1}{2}ht \,.\, \frac{1}{4}h^2 = \frac{1}{6}\,th^3$.
 $\text{I} = \frac{1}{2}bth^2 + \frac{1}{6}th^3 = (3b + h)t \,.\, \frac{1}{6}h^2.\quad r^2 = \dfrac{3b + h}{b + h} \cdot \dfrac{h^2}{12}.$

II. Hollow square, side h; axis parallel to side.
$I = \frac{2}{3}\, h^3t. \quad r^2 = \frac{1}{6}\, h^2.$

III. Do. do. axis diagonal. $y = h\sqrt{\frac{1}{2}}$.
$I = 4 \cdot \frac{1}{3}ht \quad h^2 \quad \frac{1}{2} = \frac{2}{3}\, h^3t. \quad r^2 = \frac{1}{6}h^2.$

IV. Hollow triangle, base b, sides, each a, altitude$^2 = h^2 = a^2 + \frac{1}{4}b^2$. Distance of centre of gravity from base, $y_1 = \frac{ah}{2a+b}$. Axis is parallel to base.

$$I = \tfrac{2}{3}\, ath^2 - (2a+b)ty'^2 = \tfrac{2}{3}ath^2 - \frac{a^2th^2}{2a+b}$$

$$= ath^2\left(\tfrac{2}{3} - \frac{a}{2a+b}\right).$$

V. Hollow triangle, axis through vertex, perpendicular to b.

$$I = 2 \cdot \tfrac{1}{3} \cdot \tfrac{1}{2}bt \cdot \tfrac{1}{4}b^2 + 2 \cdot \tfrac{1}{3}at \quad \tfrac{1}{4}b^2 = (2a+b)t \cdot \frac{b^2}{12}.$$

$$r^2 = \frac{b^2}{12}.$$

VI. Hollow circle, radius R. Polar moment $= 2\pi Rt \cdot R^2 = 2\pi R^3t. \quad I_z = \frac{1}{2}J = \pi R^3t.$
$r^2 = \pi R^3t \div 2\pi Rt = \frac{1}{2}R^2 = \frac{1}{8}d^2$

VII. Hollow hexagon, side a, axis through opposite vertices.

$$I = (2at + 4 \cdot \tfrac{1}{3}at)\tfrac{3}{4}a^2 = \frac{5}{2}\, a^3t. \quad r^2 = \frac{5}{12}a^2.$$

VIII. Cross of equal arms. Same as hollow square, II., III., if each arm $= h$.

IX. Angle, unequal legs, h and b, about an axis parallel to b.

$$y' \text{ from vertex of angle} = \frac{h^2}{2(b+h)}; \quad \tfrac{1}{2}h - y' = \frac{bh}{2(b+h)}.$$

$$I = bt \cdot y'^2 + \frac{1}{12}\, h^3t + ht(\tfrac{1}{2}h - y')^2$$

$$= \left(\frac{bh^4}{4(b+h)^2} + \frac{h^3}{12} + \frac{b^2h^3}{4(b+h)^2}\right)t = \frac{4b+h}{b+h} \cdot \frac{h^3t}{12}.$$

$$r^2 = \frac{4b+h}{(b+h)^2} \cdot \frac{h^3}{12}.$$

If axis is parallel to h, transpose b and h.

X. Angle, about an axis through centre of gravity of each leg. Least value of I.

$$y' \div \tfrac{1}{2}b = h \div \sqrt{(b^2+h^2)}; \quad y'^2 = \frac{b^2h^2}{4(b^2+h^2)}.$$

$$I = \tfrac{1}{3}(b+h)y'^2t. \quad r^2 = \frac{b^2h^2}{12(b^2+h^2)}.$$

XI. Angle, equal legs, h. Make $b = h$ in IX. Axis parallel to one leg. $y' = \frac{1}{4}h$.

$$I = \frac{5}{24}h^3t. \quad r^2 = \frac{5}{48}h^2.$$

XII. Do. Do. Axis through centre of gravity of each leg.

$$I = \frac{1}{12}h^3t. \quad r^2 = \frac{1}{24}h^2$$

XIII. I Beam, web h, each flange b, axis perpendicular to web.

$$I = (2b \cdot \tfrac{1}{4}h^2 + \frac{1}{12}h^3)t = \frac{6b + h}{12}h^2t.$$

$$r^2 = \frac{6b + h}{2b + h} \cdot \frac{h^2}{12}.$$

XIV. Do. Do. Axis along web, h.

$$I = 4 \cdot \tfrac{1}{3} \cdot \tfrac{1}{2}bt \cdot \tfrac{1}{4}b^2 = \tfrac{1}{6}b^3t. \quad r^2 = \tfrac{1}{6}b^3 \div (2b + h).$$

XV. Channel, web h, each flange b. Axis perpendicular to web. Same as XIII.

XVI. Do. Do. Axis parallel to web.

$$y' \text{ from back } = \frac{b^2}{2b + h}.$$

$$I = 2 \cdot \tfrac{1}{3}bt \cdot \tfrac{1}{4}b^2 + 2bt(\tfrac{1}{2}b - y')^2 + hty'^2$$

$$= \frac{2(b + h)^2 + bh}{(2b + h)^2} \cdot \frac{b^3t}{3}. \quad r^2 = \frac{2(b + h)^2 + bh}{(2b + h)^3} \cdot \tfrac{1}{3}b^3.$$

XVII. Z bar, web h, each flange b, axis perpendicular to web. Same as XIII. See Plate III., XIV.

XVIII. Do. Do. Axis along web h.

$$I = 2 \cdot \tfrac{1}{3}bt \cdot b^2 = \tfrac{2}{3}b^3t. \quad r^2 = \tfrac{2}{3} \cdot \frac{b^3}{2b + h}.$$

The sections are supposed to be very thin and the average thickness is to be used, found by dividing the area by the sum of the given lengths of lines. If r^2 alone is desired, t may be neglected.

Examples.—$4 \times 6 \times \frac{3}{8}$ L. Axis parallel to shorter leg.

$$I = \frac{16 + 6}{10} \cdot \frac{6^3 \cdot 3}{12 \cdot 8} = 14.8. \quad r^2 = \frac{14.8 \times 8}{10 \times 3} = 3.95.$$

Axis parallel to longer leg. $I = \dfrac{24 + 4}{10} \cdot \dfrac{4^3 \cdot 3}{12 \cdot 8} = 5.6.$

10 in. I beam, 5 in. flange, area 9.7 sq. in. $2b + h = 20 \therefore t = 0.5$. $I = \dfrac{30 + 10}{12} \cdot 100 \cdot \dfrac{5}{10} = 167.$

XIX. Circular Arc, axis through centre parallel to chord. Fig. 48.

Length of arc $2R\theta$.

$ds : dx = R : y. \quad \therefore yds = Rdx.$

I about A B $= t\int y^2 ds = Rt\int ydx = Rt$. area A B C D

$$= R^3(\theta + \sin\theta\cos\theta)t,$$

since area $= 2R\theta \cdot \frac{1}{2}R + 2 \cdot \frac{1}{2}R\sin\theta \cdot R\cos\theta$.

Distance of centre of gravity of arc from A B, $y' = \dfrac{\int yds}{\int ds}$

$$= \frac{R\int dx}{\int ds} = \frac{R \,.\, \text{chord C D}}{\text{arc C D}} = \frac{R\sin\theta}{\theta}.$$

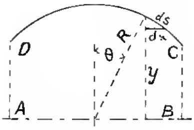

Fig. 48.

$$I = R^3t(\theta + \sin\theta\cos\theta) - 2R\theta t \; y'^2$$

$$= R^3\left(\theta + \sin\theta\cos\theta - 2\frac{\sin^2\theta}{\theta}\right)t.$$

$$S = 2R\theta t; \text{ If } b = 2R\sin\theta,$$

$$r^2 = R^2\left(\frac{1}{2} + \frac{\sin\theta\cos\theta}{2\theta} - \frac{\sin^2\theta}{\theta^2}\right)$$

$$= \frac{b^2}{8}\left(\frac{1}{\sin^2\theta} + \frac{\cos\theta}{\theta\sin\theta} - \frac{2}{\theta^2}\right).$$

XX. Do. Do. Axis through centre, perpendicular to chord. Fig. 49.

$$I = t\int y^2 ds = Rt\int ydx = Rt \text{ . area segment}$$

$$= R^3(\theta - \sin\theta\cos\theta)t. \quad S = 2R\theta t. \quad \text{If } b = 2R\sin\theta,$$

$$r^2 = R^2\left(\frac{1}{2} - \frac{\sin\theta\cos\theta}{2\theta}\right) = \frac{b^2}{8\sin^2\theta}\left(1 - \frac{\sin 2\theta}{2\theta}\right)$$

100. **Spacing of Channels.**—To find the distance d which should separate two channels, so that r^2 may be the same about both rectangular axes.

1st. When the flanges are turned out. Notation as in XVI. Axis parallel to web. For both channels, distance apart being d,

$$I = 2ht \cdot \tfrac{1}{4}d^2 + 4 \cdot \tfrac{1}{3}(b + \tfrac{1}{2}d)^3t - 4 \cdot \tfrac{1}{3} \cdot \tfrac{1}{8}d^3t$$
$$= \tfrac{1}{2}(h + 2b)d^2t + \tfrac{1}{3}(4b + 6d)b^2t.$$

Axis perpendicular to web, by XV,

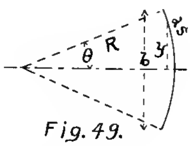

Fig. 49.

$$I = 2\,\frac{6b + h}{12} \cdot h^2t.$$

Equate these two values and transpose

$(h + 2b)d^2 + 4b^2d = \frac{1}{3}h^3 + 2bh^2 - \frac{8}{3}b^3$. Solve for d.

2d. When the flanges are turned in.

$$I = 2ht \cdot \tfrac{1}{4}d^2 + 4 \cdot \tfrac{1}{3} \cdot \tfrac{1}{8}d^3t - 4 \cdot \tfrac{1}{3}(\tfrac{1}{2}d - b)^3t$$
$$= \tfrac{1}{2}(h + 2b)d^2t + \tfrac{1}{3}(4b - 6d)b^2t.$$
$$\therefore (h + 2b)d^2 - 4b^2d = \tfrac{1}{3}h^3 + 2bh^2 - \tfrac{8}{3}b^3.$$

Example.—7 in. channel, 2 in. flanges; flanges turned out. $11d^2 + 16d = 289$; $d = 4\frac{1}{2}$ in.—.

Flanges turned in. $11d^2 - 16d = 289$; $d = 6$ in.—.

Examples.—1. Find the moment of inertia of a trapezoid, bases a and b, height h, about one base.

2. A 12 in. joist has two mortises cut through it, VI., Plate II., each 2 in. square, and 2 in. from edge of joist to edge of mortise. How much is that section of the joist weakened? $\frac{7}{27}$ or 26%.

3. A bridge floor is made of plates rolled to half hexagon troughs, X., Plate III., 6 in. face, 5.2 in. deep, 12 in. opening, $\frac{1}{2}$ in. thick. Find the resisting moment of a section 18 in. wide. $20.8f$.

4. If that floor is 14 ft. between trusses and carries two rails, 5 ft. apart, each loaded with 2,000 lbs. per running foot, what will be the unit stress? 7,790 lbs.

5. Six thin rolled shapes, web a, make a hexagonal column, radius a, with riveted outside flanges, each b in width. Prove that

$$r^2 = \frac{a^3 + 4(a + b)^3}{12(a + 2b)}.$$

CHAPTER VI.

FLEXURE AND DEFLECTION OF SIMPLE BEAMS.

101. **Introduction.**—As the stresses of tension and compression which make up the resisting moment at any section of a beam cause elongation and shortening of the respective longitudinal elements or layers on either side of the neutral plane, a curvature of the beam will result. This curvature will be found to depend upon the material used for the beam, upon the magnitude and distribution of the load, the span of the beam and manner of support, and upon the dimensions and form of cross-section. It is at times desirable to ascertain the amount of *deflection*, or perpendicular displacement from its original position, of any point, or of the most displaced point, of any given beam carrying a given load.

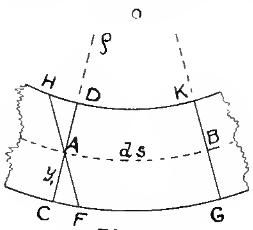

Fig. 50

Further, the investigation of the forces and moments which act on beams supported in any other than the ways already discussed requires the use of equations that take account of the bending of the beams under these moments. There are too many unknown quantities to admit of a solution by the principles of statics alone. The required equations involve expressions for the inclination or *slope* of the tangent to the curved neutral axis of the bent beam at any point, and its *deflection*, or perpendicular displacement, at any point from its original straight line, or from a given axis.

102. **Formula for Curvature.**—If, through the points A and B, on the neutral axis of a bent beam, Fig. 50, and distant ds apart, normals C D and K G to the curve of this neutral axis are drawn, the distance from A B to their intersection will be the radius of curvature ρ for that portion of the curve. If through A a plane F H is passed parallel to

K G, the distance F C will be the elongation, or H D will be the shortening, from the unit stress f, of the extreme fibre which was ds long before flexure. Cross-sections plane before flexure are plane after flexure, § 75.

A O $= \rho$; A C $= y_1$; C F $= \frac{f}{E} ds$, § 10. From similar triangles A C F and O A B, $\rho : ds = y_1 : \frac{f}{E} ds$, or $\rho = \frac{E y_1}{f}$. As, by § 77, $M = f I \div y_1$, $\frac{1}{\rho} = \frac{M}{E I}$,

the reciprocal of the radius of curvature, called the *curvature* or the amount of bending at any one point.

103. **Slope and Deflection.**—If the curve of the neutral axis is referred to rectangular co-ordinates, x being parallel to the original straight axis of the beam, and v being perpendicular to the same, the differential calculus gives for the radius of curvature, $\rho = \frac{ds^3}{dx d^2 v}$. For very slight curvature, such as is found in practical, safe beams, ds along the curve may be assumed equal to dx along the axis of x. Then

$$\frac{1}{\rho} = \frac{d^2 v}{dx^2} = \frac{M}{E I}.$$

As M is a function of x, as has been seen already, the first definite integral, $\frac{dv}{dx}$, will give the tangent of the inclination or the *slope* of the tangent to the curve of the neutral axis at any point x, and the second integral will give v, the *deflection*, or perpendicular ordinate to the curve from the axis of x.

While the following applications of the operations indicated in the last paragraph are examples only, the results in most of them will be serviceable for reference.

The student must be careful, in solving problems of this class, to use a *general value* for M, and *not M maximum.* The origin of co-ordinates will be taken at a point of support, such a point being definitely located; x is measured horizontally, v vertically, and $-v$ denotes deflection downwards.

The greatest deflection for a given load is v max. The greatest allowable deflection for a fibre stress f is v_1.

If M ÷ I is constant, the beam bends to the arc of a circle. This happens where M is constant and I is constant, or where I varies as M.

Example.—The middle segment of a uniform beam under § 69, case XII. If W = 2,000 lbs., a = 5 ft. = 60 in. I = $\frac{1}{12}bh^3 = 6 \cdot 12^3 \div 12 = 864$, and E = 1,400,000,

$$\frac{1}{\rho} = \frac{120,000}{864 \cdot 1,400,000} = \frac{1}{10,080} \text{ and } \rho = 10,080 \text{ in.} = 840 \text{ ft.}$$

APPLICATIONS:—

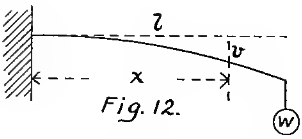

Fig. 12.

$$\text{M max.} = -\text{W}l.$$
$$v \text{ max.} = -\frac{\text{W}l^3}{3\text{ E I}}.$$
$$v_1 = \frac{fl^2}{3\text{E}y_1}.$$

104. **Beam fixed at one end;** single load at the other; origin at the wall; length = l; x measured horizontally, v vertically.

$$\text{M}_x = -\text{W}(l - x);\ \frac{d^2v}{dx^2} = -\frac{\text{W}}{\text{E I}}(l - x). \quad \text{Let } \frac{\text{W}}{\text{E I}} = \text{A}.$$

Tan. inclination, or slope, at $x = \frac{dv}{dx} = -\text{A}\int(l - x)\,dx = -\text{A}(lx - \frac{1}{2}x^2 + \text{C})$.

At the point where $x = 0$, the slope is zero, and therefore C = 0.

$$v_x = -\text{A}\int(lx - \tfrac{1}{2}x^2)dx = -\text{A}(\tfrac{1}{2}lx^2 - \tfrac{1}{6}x^3 + \text{C}^1)$$
$$v = 0, \text{ when } x = 0, \quad \therefore \quad \text{C}^1 = 0.$$

For $x = l$, tan i, or max. slope $= -\text{A}(l^2 - \frac{1}{2}l^2) = -\frac{\text{W}l^2}{2\text{ E I}}$;

and v max., or max. deflection $= -\text{A}(\frac{1}{2}l^3 - \frac{1}{6}l^3) = -\frac{\text{W}l^3}{3\text{ E I}}$.

To determine the maximum allowable deflection of a given beam consistent with a safe unit stress in the extreme fibre at the section of maximum bending moment, substitute, in the expression for v max., the value of W in terms of f.

Thus, by § 77, M max. $= -\text{W}l = \frac{f\text{I}}{y_1}$. $\therefore$

$$\text{W} = -\frac{f\text{I}}{y_1 l}; \text{ and } v_1 = -\frac{\text{W}l^3}{3\text{ E I}} = \frac{fl^2}{3\text{E}y_1}.$$

Example.—If $l = 60$ in., $b = 4$ in., $h = 8$ in., W = 800 lbs., E $= 1,400,000$; $I = \frac{4 \cdot 8^3}{12} = \frac{512}{3}$; max. slope $= \frac{800 \cdot 60^2 \cdot 3}{2 \cdot 1,400,000 \cdot 512}$ $= 0.006$; v max. $= \frac{800 \cdot 60^3 \cdot 3}{3 \cdot 1,400,000 \cdot 512} = 0.24$ in.; and, if f $= 1,200$ lbs., max. safe deflection $= \frac{1,200 \times 60^2}{3 \times 1,400,000 \times 4}$ $= 0.26$ in.*

It will be seen that, for a given weight, the maximum bending moment varies as the length l; the maximum slope varies as l^2; and the maximum deflection as l^3. The slope and deflection also vary inversely as I, or inversely as the breadth and the cube of the depth of the beam. The maximum safe deflection, however, consistent with the working unit stress f, varies as l^2, and inversely as y_1, or the depth of the beam. These relationships are true for other cases, as will be seen in what follows.

The ease with which problems regarding deflection are solved depends greatly upon the point taken for the origin, as it influences the value of the constants of integration.

$$\text{M max.} = -(wl)\ \tfrac{1}{2}l = -\tfrac{1}{2}wl^2.$$

$$v \text{ max.} = -\frac{(wl)l^3}{8\text{EI}}. \quad v_1 = \frac{fl^2}{4\text{E}y_1}.$$

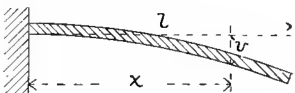

Fig. 13.

105. Beam fixed at one end; uniform load of w per unit over the whole length l; origin at the wall.

$$\text{M}_x = -\tfrac{1}{2}w(l - x)^2; \ \frac{d^2v}{dx^2} = -\frac{w}{2\text{EI}}(l^2 - 2lx + x^2).$$

$$\text{Let B} = \frac{w}{2\text{EI}}.$$

$$\text{Slope at } x = \frac{dv}{dx} = -\text{B}\int(l^2 - 2lx + x^2) =$$

$$-\text{B}(l^2x - lx^2 + \tfrac{1}{3}x^3 + \text{C}).$$

$$\text{When } x = 0, \ \frac{dv}{dx} = 0; \ \therefore \text{C} = 0.$$

$$v_x = -\text{B}(\tfrac{1}{2}l^2x^2 - \tfrac{1}{3}lx^3 + \tfrac{1}{12}x^4 + \text{C}').$$

$$\text{When } x = 0, x = 0; \ \therefore \text{C}' = 0.$$

*Cancel factors before reducing.

For $x = l$, tan. max. slope $= -\text{B}(l^3 - l^3 + \frac{1}{3}l^3) = -\frac{(wl)l^2}{6\text{EI}}$;

$$\text{and } v \text{ max.} = -\text{B}(\tfrac{1}{2}l^4 - \tfrac{1}{3}l^4 + \tfrac{1}{12}l^4) = -\frac{(wl)l^3}{8\text{EI}}.$$

Again, for maximum safe deflection, consistent with unit stress f in the extreme fibre at the dangerous section,

$$\text{M max.} = -(wl)\,\tfrac{1}{2}l = \frac{f\text{I}}{y_1}.$$

$$\therefore\ wl = -\frac{2f\text{I}}{y_1 l};\ \text{and } v_1 = -\frac{(wl)l^3}{8\text{EI}} = \frac{fl^2}{4\text{E}y_1}.$$

106. Combination of Uniform Load and Single Load at one end of a beam fixed at the other end. Add the corresponding values of the two cases preceding.

$$\text{M max.} = -[\text{W}l + \tfrac{1}{2}(wl)l];$$

$$\text{tan. max. slope} = -\frac{1}{\text{EI}}\,[\tfrac{1}{2}\,\text{W}l^2 + \tfrac{1}{6}\,(wl)l^2];$$

$$v \text{ max.} = -\frac{1}{\text{EI}}\,[\tfrac{1}{3}\text{W}l^3 + \tfrac{1}{8}(wl)l^3] = -\frac{l^3}{3\text{EI}}\,(\text{W} + \tfrac{3}{8}wl).$$

Note, in the expression for v max., the relative deflections due to a load at the end and to the same load distributed along the beam; and compare with the respective maximum bending moments.

Example.—If the preceding beam weighs 50 lbs., the additional deflection will be $\frac{50\,.\,60^3\,.\,3}{8\,.\,1{,}400{,}000\,.\,512} = 0.005$ in., too small a quantity to be of importance. In the majority of cases, the weight of the beam itself may be neglected, unless the span is long.

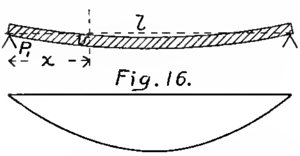

Fig. 16.

$$\text{M max.} = \tfrac{1}{8}(wl)l.$$

$$v \text{ max.} = -\frac{5}{8.\ 48}\cdot\frac{(wl)l^3}{\text{EI}}$$

$$v_1 = \frac{5}{48}\cdot\frac{fl^2}{\text{E}y_1}.$$

107. Beam Supported at Both Ends; uniform load of w per unit over the whole length l; origin at left point of support. By § 69, VI.

$$\text{M}_\text{x} = \tfrac{1}{2}w(l - x)x;\ \frac{d^2v}{dx^2} = \frac{w}{2\text{E I}}\,(lx - x^2) = \text{B}\,(lx - x^2).$$

$$\frac{dv}{dx} = \text{B}(\tfrac{1}{2}lx^2 - \tfrac{1}{3}x^3 + \text{C}) = \text{tan. slope at } x.$$

To determine the constant of integration, we see from Fig. 16 that $\frac{dv}{dx} = 0$, when $x = \frac{1}{2}l$. Then

$$0 = \tfrac{1}{8}l^3 - \tfrac{1}{24}l^3 + C, \text{ or } C = -\tfrac{1}{12}l^3. \quad \therefore$$

$$\frac{dv}{dx} = B(\tfrac{1}{4}lx^2 - \tfrac{1}{6}x^3 - \tfrac{1}{12}l^3).$$

$$v_x = B(\tfrac{1}{6}lx^3 - \tfrac{1}{12}x^4 - \tfrac{1}{12}l^3x + C').$$

As $v = 0$, when $x = 0$; $C' = 0$, and disappears.

For $x = 0$, or $x = l$, tan. max. slope $= \pm \frac{(wl)l^2}{24EI}$,

the opposite signs denoting opposite slopes at the two ends.

$$v \text{ max. (when } x = \tfrac{1}{2}l) = \frac{w}{2EI}\left(\frac{l^4}{48} - \frac{l^4}{192} - \frac{l^4}{24}\right) = -\frac{5}{384}\,\frac{(wl)l^3}{EI}.$$

For maximum safe deflection, consistent with a unit stress f in the extreme fibre at the middle section,

$$\frac{fI}{y_1} = \tfrac{1}{8}(wl)l; \quad \therefore wl = \frac{8fI}{y_1 l}, \text{ and } v_1 = \frac{5}{48} \cdot \frac{fl^2}{Ey_1}.$$

Examples.—A pine floor-joist, uniformly loaded, section 2×12 inches, span 14 ft. = 168 in., has deflected $\frac{3}{4}$ in. at the middle. Is it safe? E = 1,500,000. By the last formula,

$$\tfrac{3}{4} = \frac{5}{48}f\,\frac{14^2 \cdot 12^2}{1\,500{,}000 \quad 6}; \; f = 2{,}300 \text{ lbs.}$$

and the beam is overloaded.

What weight is it carrying? By formula for v max.

$$\tfrac{3}{4} = \frac{5}{384}\,wl\,\frac{14^3 \cdot 12^3 \cdot 12}{1{,}500{,}000 \cdot 2 \cdot 12^3}; \; wl = 5{,}248 \text{ lbs.}$$

$$M \text{ max.} = \tfrac{1}{4}Wl.$$

$$v \text{ max.} = \frac{-Wl^3}{48EI}. \qquad v_1 = \tfrac{1}{12}\,\frac{fl^2}{Ey_1}.$$

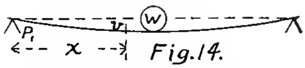

Fig. 14.

108. Beam Supported at Both Ends; single load W at middle of span l; origin at left point of support.

$$M_x = \tfrac{1}{2}Wx; \quad \frac{d^2v}{dx^2} = \frac{W}{2EI}x = Ax$$

This expression will apply only from $x = 0$ to $x = \frac{1}{2}l$; but, as the two halves of the deflection curve are symmetrical, the discussion of the left half will suffice.

$$\frac{dv}{dx} = A(\tfrac{1}{2}x^2 + C);\ \frac{dv}{dx} = 0,\ \text{when } x = \tfrac{1}{2}l;\ \therefore\ C = -\tfrac{1}{8}l^2.$$

$$\frac{dv}{dx} = A(\tfrac{1}{2}x^2 - \tfrac{1}{8}l^2).$$

$$v_x = A(\tfrac{1}{6}x^3 - \tfrac{1}{8}l^2x + C');\ v = 0,\ \text{when } x = 0;\ \therefore\ C' = 0.$$

For $x = 0$, tan. max. slope $= -\dfrac{Wl^2}{16EI}$. The limit $x = l$ is not applicable.

$$v \text{ max. (when } x = \tfrac{1}{2}l) = \frac{W}{2EI}\left(\frac{l^3}{48} - \frac{l^3}{16}\right) = -\frac{Wl^3}{48EI}.$$

For max. safe deflection, since $\frac{1}{4}Wl = \dfrac{fI}{y_1}$, as before,

$$W = \frac{4fI}{y_1 l},\ \text{and}\ v_1 = \frac{fl^2}{12Ey_1}.$$

Notice the numerical coefficients of v max. in §§ 104, 105, 108, and 107. They are $\frac{1}{3}$, $\frac{1}{8}$, $\dfrac{1}{48}$ and $\dfrac{5}{8 \cdot 48}$. M max. varies as 1, $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}Wl$.

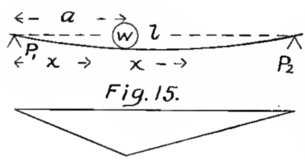

Fig. 15.

Example.—A 10 in. steel I beam of 33 lbs. per ft. and I = 162, span, 15 ft., = 180 in., carries in addition a uniform load of 767 lbs. per ft. of span and 6,000 lbs. concentrated at the middle. What will be its deflection and the max. unit fibre stress? From §§ 107 and 108,

$$v \text{ max.} = \left(\frac{5 \cdot 800 \cdot 15}{384} + \frac{6{,}000}{48}\right)\frac{180^3}{29{,}000{,}000 \cdot 162} = 0.35 \text{ in.}$$

From § 69, IV. and VI., $\dfrac{f \cdot 162}{5} = \left(\dfrac{800 \cdot 15}{8} + \dfrac{6{,}000}{4}\right)180;$

$$f = 16{,}667 \text{ lbs.}$$

109. Single Weight on Beam of Span l, supported at both ends; W at a given distance a from the origin, which is at the left point of support.

As the load is eccentric, the curve of the beam is unsymmetrical, and equations must be written for each portion, $x < a$ and $x > a$.

ON LEFT OF WEIGHT.

$$P_1 = W\frac{l-a}{l}; \; M_X = \frac{W(l-a)}{l}x.$$

$$\frac{d^2v}{dx^2} = \frac{W}{EIl}(l-a)x = A(l-a)x.$$

$$\frac{dv}{dx} = A(\tfrac{1}{2}lx^2 - \tfrac{1}{2}ax^2 + C). \qquad (1.)$$

$$v_X = A(\tfrac{1}{6}lx^3 - \tfrac{1}{6}ax^3 + Cx + C'')$$

$v_X = 0$, when $x = 0$; $\therefore C'' = 0$.

ON RIGHT OF WEIGHT.

$$P_2 = \frac{Wa}{l}; \; M_X = \frac{Wa}{l}(l-x).$$

$$\frac{d^2v}{dx^2} = \frac{W}{EIl}(al - ax) = A(al - ax).$$

$$\frac{dv}{dx} = A(alx - \tfrac{1}{2}ax^2 + C'). \qquad (2.)$$

$$v_X = A(\tfrac{1}{2}alx^2 - \tfrac{1}{6}ax^3 + C'x + C''')$$

$v_X = 0$, when $x = l$; $\therefore C''' = -\tfrac{1}{3}al^3 - C'l.$

For equations to determine the constants C and C′, use the value $x = a$; when it will be evident that $\frac{dv}{dx}$ for the left segment must give the same value as does $\frac{dv}{dx}$ for the right segment; and v at a must be the same when obtained from the left column as when obtained from the right. Therefore, from (1.) and (2.),

$\tfrac{1}{2}a^2l - \tfrac{1}{2}a^3 + C = a^2l - \tfrac{1}{2}a^3 + C'$; or $C = C' + \tfrac{1}{2}a^2l$. v at a then gives $\tfrac{1}{6}a^3l - \tfrac{1}{6}a^4 + C'a + \tfrac{1}{2}a^3l = \tfrac{1}{2}a^3l - \tfrac{1}{6}a^4 + C'a - \tfrac{1}{3}al^3 - C'l$, or $C' = -\tfrac{1}{6}a^3 - \tfrac{1}{3}al^2$. Therefore $C''' = \tfrac{1}{6}a^3l$, and $C = \tfrac{1}{2}a^2l - \tfrac{1}{6}a^3 - \tfrac{1}{3}al^2$. Substitute above.

$$\frac{dv}{dx} = \frac{W}{EIl}\left(\frac{l-a}{2}x^2 + \frac{a^2l}{2} - \frac{a^3}{6} - \frac{al^2}{3}\right)$$

$$v_X = \frac{W}{EIl}\left(\frac{l-a}{6}x^3 + \frac{a^2lx}{2} - \frac{a^3x}{6} - \frac{al^2x}{3}\right)$$

$$x = 0, \text{tan. max. slope} = \frac{Wa}{EIl}\left(\frac{al}{2} - \frac{a^2}{6} - \frac{l^2}{3}\right)$$

$$= -\frac{Wa\,(l-a)\,(2l-a)}{6l\,EI}.$$

$$\frac{dv}{dx} = -\frac{W}{EIl}\left(alx - \frac{ax^2}{2} - \frac{a^3}{6} - \frac{al^2}{3}\right)$$

$$v_X = \frac{W}{EIl}\left(\frac{alx^2}{2} - \frac{ax^3}{6} - \frac{a^3x}{6} - \frac{al^2x}{3} + \frac{a^3l}{6}\right)$$

$$= \frac{Wa}{6\,EIl}(l-x)\,[a^2 - (2l-x)\,x].$$

$$x = l, \text{tan. max slope} = \frac{Wa\,(l-a)(l+a)}{6l\,EI}.$$

As a is assumed to be less than $\tfrac{1}{2}l$, and the substitution of $x = a$ in the value of $\frac{dv}{dx}$ gives a slope which is negative, the point of v max. will be found on the right of W, and for that value of x which makes $\frac{dv}{dx}$ on the right zero. Hence

$$alx - \tfrac{1}{2}ax^2 = \tfrac{1}{6}a^3 + \tfrac{1}{3}al^2.$$

$$x^2 - 2lx + l^2 = l^2 - \tfrac{1}{3}a^2 - \tfrac{2}{3}l^2 = \tfrac{1}{3}(l^2 - a^2)$$

$$l - x = \sqrt{\tfrac{1}{3}(l^2 - a^2)};$$

which is the distance from the right point of support. Sub-

stitute in the expression for v_x on the right, to obtain the maximum deflection.

It should be noticed that, when the weight is eccentric, the point of maximum deflection is found between the weight and the mid-span, and not at the point of maximum bending moment, which latter is under the weight.

110. **Two Equal Weights on Beam of Span l,** supported at the ends; each W, symmetrically placed, distant a from one end. Fig. 18.

This case may be solved by itself, but can be more readily treated by reference to § 109. Thus the maximum deflection will be at the middle, and can be found by making $x = \frac{1}{2}l$ in the above value of v for the right segment and doubling the result. Then

$$v \text{ max.} = \frac{Wa}{3\,EIl} \cdot \frac{l}{2}\left(a^2 - \tfrac{3}{4}l^2\right) = -\frac{Wa}{24\,EI}\left(3l^2 - 4a^2\right).$$

The deflection under a weight will be given by the addition of v at a and v at $(l - a)$ of the preceding case. Thus

$$v_a = -\frac{Wa^2}{3EIl}(l-a)^2;\quad v_{l-a} = -\frac{Wa^2}{6EIl}(l^2 - 2a^2).$$

$$v \text{ at } W = -\frac{Wa^2}{6EI}(3l - 4a).$$

Example.—A round iron bar, 12 ft. long, and 2 in. diam., carries two weights of 200 lbs. each at points 3 ft. distant from either of the two supported ends. The deflection at a weight $= \frac{200 \cdot 36^2 \cdot 7 \cdot 4}{6 \cdot 28{,}000{,}000 \cdot 22}(3 \cdot 12^2 - 4 \cdot 3 \cdot 12) = 0.56$ in. The maximum unit bending stress is $\frac{200 \cdot 3 \cdot 12 \cdot 7 \cdot 4}{22} = 9{,}160$ lbs.

DEFLECTION OF BEAMS OF UNIFORM STRENGTH.

It will be apparent that a beam of uniform strength will not be so stiff as a corresponding beam of uniform section sufficient to carry safely the maximum bending moment; for the stiffness arising from the additional material in the second case is lost.

111. Uniform Strength and Uniform Depth.—Since $M = nfbh^2$ and varies as bh^2, and $I = n'bh^3$ and varies as bh^3; $M \div I$ varies as $1 \div h$. But if h is constant, $M \div I$ is constant and $\dfrac{d^2v}{dx^2}$ is constant. Therefore all beams of this class bend to the arc of a circle.

I. Beam fixed at one end only, and loaded with W at the other. § 84, Fig. 27.

$$\frac{d^2v}{dx^2} = \frac{M}{E\,I} = -\frac{W}{E\,I}(l - x).$$

If, in all cases, I_0 = moment of inertia at the largest section, which is in this case at the wall, I at the distance x from the wall will be to I_0 as $l - x$ is to l, or $I = I_0 \dfrac{l - x}{l}$. Therefore

$$\frac{d^2v}{dx^2} = -\frac{Wl}{E\,I_0},$$ a constant, as stated above.

Note that the quantity divided by $E\,I_0$ is M max.

$$\frac{dv}{dx} = -\frac{Wl}{E\,I_0}\int_o^x dx = -\frac{Wl}{E\,I_0}x;$$

$$v \text{ max.} = -\frac{Wl}{E\,I_0}\int_o^l x\,dx = -\frac{Wl^3}{2E\,I_0},$$

a deflection 50% in excess of that of the corresponding uniform beam, while the max. slope is twice as great.

Examples.—If a triangular sheet of metal, like the dotted triangle in Fig. 51, is cut into strips, as represented by the dotted lines, and these strips are superimposed as shown above, the strips, if fastened at the ends, and subjected to W as shown, will tend to bend in arcs of circles, and will remain approximately in contact. If l = 10 in., b = 4 in., $h = \frac{1}{4}$ in., W = 400 lbs. and E = 28,000,000, the deflection will be $\dfrac{400 \,.\, 10^3 \,.\, 4^3 \,.\, 12}{2 \,.\, 28{,}000{,}000 \,.\, 4} = 1.37$ in.

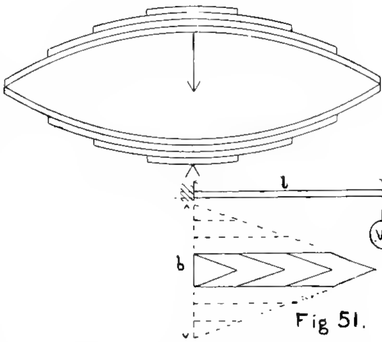

Fig 51.

An elliptical steel spring 2 ft. long, of 4 layers as shown, each 2 in. broad and ⅛ in. thick, under a load of 100 lbs. at its

middle, will, if $E = 29{,}000{,}000$, deflect $\frac{100 \cdot 12^3 \cdot 8^3 \cdot 12}{2 \cdot 29{,}000{,}000 \cdot 8} = 2.3$ in. The max. unit fibre stress will be $\frac{50 \cdot 12 \cdot 6 \cdot 8^2}{8} = 28{,}800$ lbs. Note that one-half of the weight is found at each hinge, and that the deflection of one arm is doubled by the use of two springs as shown.

II. Beam fixed at one end only and uniformly loaded with w per unit. § 84, Fig 29.

$$\frac{d^2v}{dx^2} = -\frac{w}{2\mathrm{E\,I}}(l-x)^2. \quad \mathrm{I} : \mathrm{I}_0 = b : b_0 = (l-x)^2 : l^2.$$

$$\frac{dv}{dx} = -\frac{wl^2}{2\mathrm{E\,I}_0}\int_o^x dx = -\frac{wl^2}{2\mathrm{E\,I}_0}x;$$

$$v \text{ max.} = -\frac{wl^2}{2\mathrm{E\,I}_0}\int_o^l xdx = -\frac{wl^4}{4\mathrm{E\,I}_0},$$

a deflection twice that of the corresponding uniform beam.

In these two cases there are no constants of integration, since $\frac{dv}{dx}$ and $v = 0$, when $x = 0$.

III. Beam supported at both ends and carrying W at middle. § 84, Fig. 31.

$$\frac{d^2v}{dx^2} = \frac{\mathrm{W}x}{2\mathrm{E\,I}}; \; \mathrm{I} : \mathrm{I}_0 = x : \tfrac{1}{2}l; \; \frac{dv}{dx} = \frac{\mathrm{W}l}{4\mathrm{E\,I}_0}(x + \mathrm{C}).$$

$$\frac{dv}{dx} = 0, \text{ when } x = \tfrac{1}{2}l; \therefore \mathrm{C} = -\tfrac{1}{2}l.$$

$$v = \frac{\mathrm{W}l}{4\mathrm{E\,I}_0}(\tfrac{1}{2}x^2 - \tfrac{1}{2}lx). \quad \text{When } x = \tfrac{1}{2}l, \; v \text{ max.} = -\frac{\mathrm{W}l^3}{32\mathrm{E\,I}_0},$$

a deflection 50% greater than for a corresponding uniform beam.

IV. Beam supported at both ends, and uniformly loaded with w per unit of length. § 84, Fig. 33.

$$\frac{d^2v}{dx^2} = \frac{w}{2\mathrm{E\,I}}(lx - x^2); \; \mathrm{I} : \mathrm{I}_0 = x(l-x) : \tfrac{1}{4}l^2;$$

$$\frac{dv}{dx} = \frac{wl^2}{8\mathrm{E\,I}_0}(x + \mathrm{C}).$$

$$\frac{dv}{dx} = 0, \text{ when } x = \tfrac{1}{2}l; \; . \; . \; \mathrm{C} = -\tfrac{1}{2}l.$$

$$v = \frac{wl^2}{8\mathrm{E\,I}_0}(\tfrac{1}{2}x^2 - \tfrac{1}{2}lx). \quad \text{When } x = \tfrac{1}{2}l, \; v \text{ max.} = -\frac{wl^4}{64\mathrm{E\,I}_0},$$

a deflection 20% greater than for a corresponding uniform beam.

111 A. **Uniform Strength and Uniform Breadth.**—In these cases, as b is constant, $I : I_0 = h^3 : h^3_0$ or $I = I_0 \frac{h^3}{h^3_0}$.

V. Beam fixed at one end only, and loaded with W at the other. § 84, Fig. 28.

$$\frac{d^2v}{dx^2} = -\frac{W}{EI}(l - x) \; ; \; h^2 : h^2_0 = l - x : l, \therefore \frac{h}{h_0} = \sqrt{\left(\frac{l-x}{l}\right)}.$$

$$I = I_0 \frac{h^3}{h^3_0} = I_0 \sqrt{\left(\frac{(l-x)^3}{l^3}\right)} \; ; \; \frac{d^2v}{dx^2} = -\frac{Wl}{EI_0} l^{\frac{1}{2}} (l - x)^{-\frac{1}{2}}.$$

$$\frac{dv}{dx} = \frac{Wl^{\frac{3}{2}}}{EI_0}\left[-2(l - x)^{\frac{1}{2}} + C\right] =$$

$$-\frac{Wl^{\frac{3}{2}}}{EI_0}\left[-2(l - x)^{\frac{1}{2}} + 2l^{\frac{1}{2}}\right].$$

$$v \text{ max.} = \frac{2Wl^{\frac{3}{2}}}{EI_0}\int_0^l \left[(l - x)^{\frac{1}{2}} - l^{\frac{1}{2}}\right] dx$$

$$= \frac{2Wl^{\frac{3}{2}}}{EI_0}\left[-\tfrac{2}{3}(l - l)^{\frac{3}{2}} - l^{\frac{3}{2}} + \tfrac{2}{3}l^{\frac{3}{2}}\right] = -\frac{2}{3}\frac{Wl^3}{EI_0},$$

or twice the deflection of a corresponding uniform beam.

VI. Beam fixed at one end only and uniformly loaded with w per unit of length. § 84, Fig. 30.

$$\frac{d^2v}{dx^2} = -\frac{w}{EI}\cdot\frac{(l-x)^2}{2}; \quad \frac{h}{h_0} = \frac{l-x}{l}. \quad I = I_0\frac{(l-x)^3}{l^3}.$$

$$\frac{dv}{dx} = -\frac{wl^3}{2EI_0}\int\frac{dx}{l-x} = \frac{wl^3}{2EI_0}(\log(l-x) - \log l).$$

$$v \text{ max.} = \frac{wl^3}{2EI_0}\int_0^l (\log(l-x) - \log l)\, dx^*$$

$$= \frac{wl^3}{2EI_0}\Big[x\log(l-x) - x - l\log(l-x) - x\log l\Big]_0^l$$

$$= \frac{wl^3}{2EI_0}(l\log 0 - l - l\log 0 - l\log l + l\log l) = -\frac{wl^4}{2EI_0},$$

or four times the deflection of a corresponding uniform beam.

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* Log $(l - x) = u$; $dx = dv$; $x = v$; $du = -\frac{dx}{l-x}$ . . . $\int \log(l-x)\,dx = x\log(l-x) + \int\frac{x}{l-x}dx$. By division $\frac{x}{l-x} = -1 + \frac{l}{l-x}$; $\therefore \int\frac{x}{l-x}dx = -\int dx + l\int\frac{dx}{l-x} = -x - l\log(l-x)$.

VII. Beam supported at both ends and carrying W at middle. § 84, Fig. 32.

$$\frac{d^2v}{dx^2} = \frac{Wx}{2E\,I}\,;\quad h^2 : h^2_0 = x : \tfrac{1}{2}l,\quad \therefore \frac{h}{h_0} = \sqrt{\frac{2x}{l}}.\qquad I = I_0\sqrt{\frac{8x^3}{l^3}};$$

$$\frac{dv}{dx} = \frac{Wl^{\frac{3}{2}}}{4\sqrt{2}E\,I_0}\int x^{-\frac{1}{2}}\,dx = \frac{Wl^{\frac{3}{2}}}{4\sqrt{2}E\,I_0}\,(2x^{\frac{1}{2}} + C).$$

$$\text{when } x = \tfrac{1}{2}l,\ \frac{dv}{dx} = 0, \text{ and } C = -2\sqrt{\tfrac{1}{2}l}.$$

$$v \text{ max.} = \frac{Wl^{\frac{3}{2}}}{2\sqrt{2}E\,I_0}\int_0^{\frac{l}{2}}(x^{\frac{1}{2}} - \sqrt{\tfrac{1}{2}l})\,dx$$

$$= \frac{Wl^{\frac{3}{2}}}{2\sqrt{2}E\,I_0}\Big[\tfrac{2}{3}x^{\frac{3}{2}} - \sqrt{\tfrac{1}{2}l}\,x\Big]_0^{\frac{l}{2}}$$

$$= \frac{Wl^{\frac{3}{2}}}{2\sqrt{2}E\,I_0}\left(\frac{2}{3}\,\frac{l^{\frac{3}{2}}}{2\sqrt{2}} - \frac{l^{\frac{3}{2}}}{2\sqrt{2}}\right) = -\frac{Wl^3}{24E\,I_0},$$

or twice the deflection of a corresponding uniform beam.

VIII. Beam supported at both ends, and uniformly loaded with w per unit of length. § 84, Fig. 34.

$$\frac{d^2v}{dx^2} = \frac{w}{2E\,I}(lx - x^2)\,;\quad h^2 : h^2_0 = x\,(l - x) : \tfrac{1}{4}l^2\,;$$

$$\therefore I = I_0\frac{8\sqrt{x^3\,(l - x)^3}}{l^3}.$$

$$\frac{dv}{dx} = \frac{wl^3}{16E\,I_0}\int\frac{dx}{(lx - x^2)^{\frac{1}{2}}} = \frac{wl^3}{16E\,I_0}\left(\text{versin}^{-1}\frac{2x}{l} + C\right).$$

$$\frac{dv}{dx} = 0, \text{ when } x = \tfrac{1}{2}l\,;\quad \therefore C = -\text{versin}^{-1}\,1 = -\tfrac{1}{2}\pi.$$

$$v \text{ max.} = \frac{wl^3}{16E\,I_0}\int_0^{\frac{1}{2}l}\left(\text{versin}^{-1}\frac{2x}{l} - \tfrac{1}{2}\pi\right)dx$$

$$= \frac{wl^3}{16E\,I_0}\Big[(x - \tfrac{1}{2}l)\,\text{versin}^{-1}\frac{2x}{l} + \sqrt{(lx - x^2)} - \tfrac{1}{2}\pi x\Big]_0^{\frac{1}{2}l}$$

$$= \frac{wl^3}{16E\,I_0}\left[\frac{l}{2} - \frac{\pi}{4}l\right] = -\frac{0.2854\,wl^4}{16E\,I_0} = -\frac{0.018\,wl^4}{E\,I_0},$$

or 37% greater deflection that for a corresponding uniform beam.

Other beams might be analyzed, where both b and h varied at the same time. The method of analysis would agree with the above; but the cases are not of sufficient practical value to warrant their discussion here.

112. **Sandwich Beam.**—If a beam is made up from two materials, placed side by side, as when a plate of iron is bolted securely between two sticks of timber, the distribution of the load between the several pieces can be found from the consideration that they are compelled to deflect equally. As the spans and the longitudinal distribution of the loads are the same, the relationship $W \div E\ I = W' \div E'\ I'$ must exist, in which the symbols for one material are distinguished from those for the other by accents. If the depth is common, $W \div E\ b = W' \div E'\ b'$ Since $f = My_1 \div I$, the maximum unit stress will also vary as $W \div b$. Therefore

$$\frac{f}{E} = \frac{f'}{E'} \text{ or } f : f' = E : E'.$$

If E for timber is 1,400,000 and for steel 28,000,000, the ratio of the stresses will be $\frac{1}{20}$; and if $f = 800$ lbs. $f' = 16{,}000$ lbs. on the sq. inch.

Example.—Two 4 in. × 10 in. sticks of timber, with a $\frac{1}{4} \times 10$ in. steel plate firmly bolted between them will have a value of $M = \dfrac{(800\ .\ 8 + 16{,}000\ .\ \frac{1}{4})\ 10^2}{6} = 173{,}333$ in. lbs., the plate supplying $\frac{5}{13}$ of the amount. The combination, for a span of 10 ft., would safely carry $\dfrac{4M}{120} = 5{,}778$ lbs. load at centre, or 11,555 lbs. distributed load, in place of 3,555 or 7,110 lbs. for the timber alone.

113. **Beams of Cement with Iron Rods.**—Plates or beams of cement are used into which a sheet of wire netting or a combination of rods of iron has been built for the purpose of increasing the tensile resistance of the combination. They are known as Monier plates. The expansion and contraction of the two materials from changes of temperature are so nearly alike that heat and cold produce no ill effects. The ultimate strength of the combination lies at the yield point of the metal; for the expansion of the iron or steel above that limit is much greater than that of the encasing cement, so that the cement breaks. Rods of square section, twisted, are used, the twisted contour increasing the hold of the iron in the cement.

The iron may be computed at not above 7,500 lbs. per sq. inch, unit tension, neglecting the *tensile* resistance of the

cement. The neutral axis, when the iron is placed at one-sixth the thickness of the plate from the tension side, should be assumed to lie at three-fourths the thickness from the same side. For mortar, one part of cement to three parts of sand, six months old, E = from 4,000,000 to 5,500,000, for the combination, a value higher than for pure cement.

As to the comparative resisting power of brick laid in cement mortar, of concrete alone, and of concrete and steel combined, there may be cited the Austrian experiments of 1893 made on arches for bridges and floor surfaces. The stress at any point of any section of an arch is that due to a combination of thrust and bending moment, or the case of a strut-beam. The span was 13⅓ ft.

Brick with cement mortar:—rise, 15¾ in., thickness at crown, 6 in.; breaking load 321.5 lbs. per sq. ft.

Concrete:—rise, 15¾ in., thickness at crown, 4 in.; breaking load 737 3 lbs. per sq. ft.

Monier arch, concrete with wire netting 8 lbs. of steel per sq. ft.:—rise, 15¾ in., thickness at crown, 3⅜ in.; breaking load, 839.7 lbs. per sq. ft.

Melan arch, concrete with I beams, 1.4 lbs. of steel per sq. ft.:—rise, 11 in., thickness at crown, 3⅛ in., breaking load, 3,360 lbs. per sq. ft.

The netting serves mainly to increase the beam strength in tension; the I beams, bent to the arch form and keyed against abutment beams, carry both the thrust and bending moment at any section, while the concrete assists the I beams by lateral support.

114. **Resilience of a Beam.**—If a beam carries a single weight W, and the deflection under that weight is v_1, the external work done by that static load on the beam is $\frac{1}{2}Wv_1$. If this value of v is that which causes the maximum safe unit stress f, the quantity $\frac{1}{2}Wv_1$ is known as the *resilience* of the beam, or the *energy of the greatest shock* which the beam can bear without injury, being the product of a weight into the height from which it must fall to produce the shock in question. For a beam supported at both ends, loaded in the middle, and of rectangular section, $v_1 = \dfrac{fl^2}{6Eh}$, and $W = \dfrac{2fbh^2}{3l}$.

$$\text{Therefore, } \tfrac{1}{2}Wv_1 = \tfrac{1}{2} \cdot \tfrac{2}{3} \cdot \frac{fbh^2}{l} \quad \frac{fl^2}{6Eh} = \tfrac{1}{18} \cdot \frac{f^2}{E} \cdot bhl.$$

The allowable shock, or the resilience, is therefore proportioned to $f^2 \div E$, which is known as the *modulus of resi-*

lience of the material, and to the *volume* of the beam. These relationships hold for other sections, and beams loaded and supported differently. The above formula should not be rigorously applied to a drop test, unless f is below the yield point.

Example.—A 2in. × 2in. bar of steel, 5 ft. between supports, if $f = 16{,}000$ lbs., ought not to be subjected, from a central weight, to more than $\frac{1}{18}\,\frac{16{,}000^2\quad 2^2\,.\,60}{29{,}000{,}000} = 118$ in. lbs. of energy.

If the load is distributed over a similar beam, the deflection at each point will be v, and the total work done will be $\frac{1}{2}\int vwdx$. If w is uniform, and the beam is supported at its ends,

$$\tfrac{1}{2}\,w\int_o^l vdx = \frac{w^2}{4\mathrm{E\,I}}\int_o^l\left(\frac{lx^3}{6}-\frac{x^4}{12}-\frac{l^3x}{12}\right)dx = \frac{w^2l^5}{240\mathrm{E\,I}}.$$

As $\frac{wl^2}{8} = \frac{f\mathrm{I}}{y_1}$, and $v_1 = \frac{fl^2}{\mathrm{E}y_1}$, the above expression becomes $\frac{1}{30}(wl)v_1$, which is one-fifteenth of the value for a single load at the middle.

115. Internal Work.—The internal work done in extending and compressing the fibres of the beam must be equal to the external work $\frac{1}{2}\mathrm{W}v_1$.

Let the cross-section be constant. The unit stress at any point of a cross-section $= p$; the force on a layer $zdy = pzdy$. The elongation or shortening of a fibre unity of section and dx long by the unit stress $p = pdx \div \mathrm{E}$. The work done in stretching or shortening the volume $zdydx = \frac{1}{2}\,.\,\frac{p^2}{\mathrm{E}}\,.\,zdydx$. But $p = \frac{f}{y_1}y = \frac{\mathrm{M}}{\mathrm{I}}y$. The work done on so much of the beam as is included between two cross-sections dx apart will be

$$\tfrac{1}{2}\,.\,\frac{\mathrm{M}^2}{\mathrm{E\,I}^2}\,dx\int_{-y_1}^{+y_1} y^2zdy = \frac{\mathrm{M}^2}{2\mathrm{E\,I}}\,dx.$$

Substitute the value of M for a particular case, in terms of x, and integrate for the whole length of the beam. Thus for a beam supported at ends and loaded with W at the middle, $\mathrm{M} = \frac{1}{2}\,\mathrm{W}x$

at any point distant x from one end, for values of x between o and $\frac{1}{2}l$. Then

$$2\,\frac{M^2}{2E\,I}\int_o^{\frac{1}{2}l} dx = \frac{W^2}{4E\,I}\int_o^{\frac{1}{2}l} x^3 dx = \frac{W^2 l^3}{96E\,I}.$$

If this value is equal to the external work $\frac{1}{2}Wv_1$, there results $v_1 = \frac{Wl^3}{48E\,I}$, as it should.

Example.—A weighted wheel of 1,000 lbs. drops ½ inch by reason of a pebble in its path, at the middle of a beam, 3 in. × 12 in., 15 ft. span. If E = 1,400,000, to find f:

$$\text{External work} = 1{,}000\,.\,\tfrac{1}{2} + \tfrac{1}{2}\,.\,1{,}000v_1 = \frac{f^2}{18E}\,3\,.\,12\,.\,180.$$

$$\text{As } v_1 = \frac{f\,.\,180^2}{12E\,.\,6},\ 500\left(1 + \frac{180^2}{12\,.\,6\,.\,E}f\right) = \frac{360}{E}f^2.$$

f = 1,741 lbs. Resulting deflection = 0.56 in. Static unit stress would be 625 lbs., and v = 0.2 in. In an actual bridge the shock is distributed more or less in the floor and adjacent beams.

Examples.—1. What is the deflection at the middle of a 2 in. by 12 in. pine joist of 12 ft. = 144 in. span, supported at ends and uniformly loaded with 3,200 lbs.? E = 1,600,000. 0.27 in.

2. What is the deflection if the load is at the middle?

0.432 in.

3. Find the stiffest rectangular cross-section, bh, to be obtained from a round log of diameter d. b = ½d.

4. A 4 in. by 6 in. joist, laid flatwise on supports 10 ft. apart, is loaded with 1,000 lbs. at the middle. The deflection is found to be 0.7 in. What is E? 1,607,000.

5. What is the max. safe deflection of a 12 in. floor joist, 14 ft. span, if f = 1,200 lbs. and E 1,600,000.

0.37 in. for uniform load; 0.29 in. for load at middle.

CHAPTER VII.

RESTRAINED AND CONTINUOUS BEAMS.

116. **Restrained Beams.**—When a beam is kept from *rotating* at one or both points of support, by being built into a wall, or by the application of a moment of such a magnitude that the tangent to the curve of the neutral plane at the point of support is forced to remain in its original direction (commonly horizontal) at such point, the beam is termed *fixed* at one or both supports. The magnitude of the moment at the point of support depends upon the span, the load and its position. It is the existence of this, at present unknown, moment which calls for the application of deflection equations to the solution or such problems as those which follow, there being too many unknown quantities to permit the treatment of Chapter III.

In applying the results obtained in the following cases to actual problems, one should feel sure that the beam is definitely fixed in direction at the given point. Otherwise the values of M, F and v will only be approximately true.

117. **Beam of Span l, Carrying a Single Weight** W in the middle and supported and fixed at both ends. Origin at left support. Fig. 52.

The reactions and end moments are now unknown. The beam may be considered either as built in at its ends (as at the right in above figure), or as having an unknown couple or moment Qb applied at each point of support (as at the left), of a magnitude just sufficient to keep the tangent there horizontal.

The reaction at either end will then be $\frac{1}{2}W + Q$, while the shear between the points of support will still be $\pm \frac{1}{2}W$. For values of $x < \frac{1}{2}l$,

$$M_x = -Q(b + x) + (\tfrac{1}{2}W + Q)\,x = \tfrac{1}{2}Wx - Qb.$$

If this value is compared with that of M_x in § 108, it is seen that a *constant* subtractive or negative moment is now felt over the whole span, in combination with the usual $\frac{1}{2}Wx$.

$M_A = -\frac{1}{8}Wl;$
$M_B = \frac{1}{8}Wl;$
$M_C = -\frac{1}{8}Wl.$

$F_A = \frac{1}{2}W;$
$F_B = -\frac{1}{2}W.$

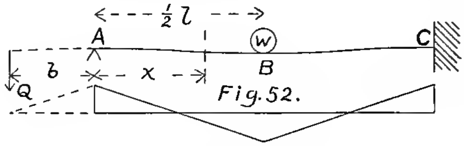


Fig. 52.

$$v \text{ max.} = \frac{Wl^3}{192 E I}; \quad v_1 = \frac{fl_2}{24Ey_1}.$$

$$\frac{d^2v}{dx^2} = \frac{1}{E I}(\tfrac{1}{2}Wx - Qb); \quad \frac{dv}{dx} = \frac{1}{E I}(\tfrac{1}{4}Wx^2 - Qbx + C).$$

$$\frac{dv}{dx} = 0, \text{ when } x = 0; \quad \therefore \quad C = 0. \quad \text{Also } \frac{dv}{dx} = 0, \text{ when } x = \tfrac{1}{2}l;$$

$$\therefore \quad 0 = \frac{Wl^2}{16} - \frac{Qbl}{2}; \text{ or } -Qb = -\tfrac{1}{8}Wl,$$

the negative bending moment at either end. That it is negative appears by making $x = 0$ in M_x above.

If this value of Qb is substituted in the first equation, giving $M_x = \frac{1}{2}W(x - \frac{1}{4}l)$ the point of contraflexure is located at $x = \frac{1}{4}l$; and the bending moment at middle, where $x = \frac{1}{2}l$, is M max. $= + \frac{1}{8}Wl$, or one-half the amount in § 108. Substitute the value of Qb in the equation for slope,

$$\frac{dv}{dx} = \frac{1}{E I}(\tfrac{1}{4}Wx^2 - \tfrac{1}{8}Wlx) = \frac{W}{4E I}(x^2 - \tfrac{1}{2}lx).$$

$$v_x = \frac{W}{4E I}(\tfrac{1}{3}x^3 - \tfrac{1}{4}lx^2 + C').$$

As $v_x = 0$, when $x = 0$; $C' = 0$. When $x = \frac{1}{2}l$,

$$v \text{ max.} = \frac{W}{4E I}\left(\frac{l^3}{24} - \frac{l^3}{16}\right) = -\frac{Wl^3}{192E I}.$$

The beam is therefore *four* times as stiff as when only supported at ends.

The slope is a maximum where $\frac{d^2v}{dx^2}$ or $M_x = 0$, that is at $x = \frac{1}{4}l$; ∴

$$\text{Tan. max. slope} = -\frac{Wl^2}{64EI}.$$

$$\text{As } \frac{fI}{y_1} = \frac{Wl}{8};\quad W = \frac{8fI}{y_1 l},\ \text{and } v_1 = \frac{fl^2}{24Ey_1};$$

so that only one-half the deflection is allowable that is permitted in § 108, but the beam may safely carry twice the load.

It is useful to notice that this beam has a bending moment at the middle equal to that which would exist there, if the beam were cut at the points of contraflexure and simply supported at those points; and that the two end segments, of length $\frac{1}{4}l$, act like two cantilevers each carrying $\frac{1}{2}W$, the shear at the point of contraflexure.

If the weight were not at the middle, the moments at the two ends would differ, equations would be needed for each of the two segments, and the solution, while possible, would be much more complicated.

Example.—A wooden beam, 6 in. square, and 7½ ft. span, is built into the wall at both ends. A central weight of 3,000 lbs. will give a max. fibre stress of $\frac{3{,}000 \cdot 90 \cdot 6}{8 \cdot 6^3} = 937\frac{1}{2}$ lbs. per sq. in. at the middle and both ends. The deflection will be $\frac{3{,}000 \cdot 90^3 \cdot 12}{192 \quad 1{,}500{,}000 \quad 6^4} = 0.07$ in., if $E = 1{,}500{,}000$. The allowable deflection, for $f = 1{,}200$, is $\frac{1{,}200 \cdot 90 \cdot 90}{24 \cdot 1{,}500{,}000 \cdot 3} = 0.09$ in., and max. allowable $W = 3{,}000 \cdot \frac{9}{7} = 3{,}860$ lbs.

118. Beam of Span l, Uniform Load of w per unit over the whole span, fixed at both ends. Origin at left support. Fig. 53.

$$M_A = -\frac{(wl)l}{12} = M_C;\quad M_B = \frac{(wl)l}{24}.$$

$$F_x = w(\tfrac{1}{2}l - x).\qquad v\ \text{max.} = \frac{(wl)l^3}{384EI}.\qquad v_1 = \frac{fl^2}{32Ey_1}.$$

As in the previous case, the reaction at either end may be represented by $\frac{1}{2}wl + Q$. The shear at x is $\frac{1}{2}wl - wx$, which expression changes sign at the middle and at either point of support; hence at those places will be found M max.

$$M_x = (\tfrac{1}{2}wl + Q)\,x - \tfrac{1}{2}wx^2 - Q(b + x) = \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2 - Qb.$$

Compare with § 107.

$$\frac{d^2v}{dx^2} = \frac{1}{EI}(\tfrac{1}{2}wlx - \tfrac{1}{2}wx^2 - Qb);$$

$$\frac{dv}{dx} = \frac{1}{EI}(\tfrac{1}{4}wlx^2 - \tfrac{1}{6}wx^3 - Qbx + C).$$

$\frac{dv}{dx} = 0$, when $x = 0$; $\therefore$ $C = 0$. $\frac{dv}{dx} = 0$, when $x = l$; $\therefore$

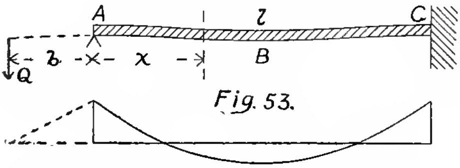


Fig. 53.

$$\frac{wl^3}{4} - \frac{wl^3}{6} - Qbl = 0;$$

$$-Qb = -\frac{wl^2}{12},$$

the negative moment at each point of support. If $x = \frac{1}{2}l$,

$$\text{M at middle} = wl^2\left(\frac{1}{4} - \frac{1}{8} - \frac{1}{12}\right) = \frac{wl^2}{24}.$$

Substitute the value of Qb, and get

$$M_x = -\tfrac{1}{2}w(x^2 - lx + \tfrac{1}{6}l^2);$$

$$\frac{dv}{dx} = -\frac{w}{2EI}(\tfrac{1}{3}x^3 - \tfrac{1}{2}lx^2 + \tfrac{1}{6}l^2x);$$

$$v = -\frac{w}{2EI}\left(\frac{x^4}{12} - \frac{lx^3}{6} + \frac{l^2x^2}{12} + C'\right).$$

Since $v = 0$, when $x = 0$, $C' = 0$. When $x = \frac{1}{2}l$,

$$v \text{ max.} = -\frac{w}{2EI}\left(\frac{l^4}{192} - \frac{l^4}{48} + \frac{l^4}{48}\right) = -\frac{(wl)l^3}{384EI},$$

which is one-fifth the value of § 107.

The points of contraflexure occur where $M_x = 0$; $\therefore$

$$x^2 - lx + \tfrac{1}{6}l^2 = 0;\quad x = \tfrac{1}{2}l \pm \tfrac{1}{2}l\sqrt{\tfrac{1}{3}}.$$

The second term is the distance from the middle, each way to the points of contraflexure. If M is calculated for the

middle point of a span $l \div \sqrt{3}$, it will prove to be $wl^2 \div 24$, as above. Since

$$\frac{(wl)l}{12} = \frac{f\,\mathrm{I}}{y_1},\ wl = \frac{12 f\,\mathrm{I}}{y_1 l};\ \text{and}\ v_1 = \frac{fl^2}{32\mathrm{E}y_1},$$

or 0.3 as much as in § 107. The beam may safely carry, however, fifty per cent. more load.

Example.—An 8 in. I beam, of 12 ft. span, carrying 1,000 lbs. per ft., if firmly fixed at both ends, and not to have a larger unit stress than 12,000 lbs., should have a value of I $= \frac{1{,}000 \cdot 12 \cdot 12 \cdot 12 \cdot 4}{12 \cdot 12{,}000} = 48$. An 8 in. steel beam, 18 lbs. to the foot, I = 57.8, will satisfy the requirement, the load then being 1,018 lbs. per ft. The deflection will be 0.6 in.

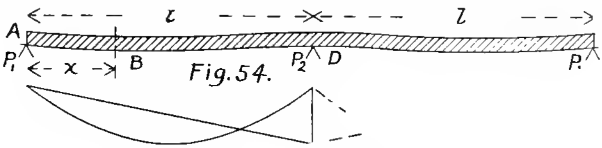


Fig. 54.

$P_1 = \frac{3}{8}wl.$ $P_2 = \frac{5}{4}wl.$ $M_B = \frac{9}{128}wl^2.$ $M_D = -\frac{1}{8}wl^2.$

119. Beam of Span l, fixed or horizontal at P_2, supported at P_1 and carrying a uniform load of w per unit, Fig. 54. Origin at P_1 and reaction unknown.

It will be seen from the sketch that a beam of length $2l$, resting upon three equidistant supports in the same straight line, will come under this case.

$$M_x = P_1 x - \tfrac{1}{2}wx^2;\quad \frac{dv}{dx} = \frac{1}{\mathrm{E\,I}}\left(\tfrac{1}{2}P_1x^2 - \tfrac{1}{6}wx^3 + C\right).$$

$$\frac{dv}{dx} = 0,\ \text{when}\ x = l;\ \therefore\ C = \tfrac{1}{6}wl^3 - \tfrac{1}{2}P_1l^2$$

$$v = \frac{1}{\mathrm{E\,I}}\left(\frac{P_1x^3}{6} - \frac{wx^4}{24} + \frac{wl^3x}{6} - \frac{P_1l^2x}{2} + C'\right).$$

$v = 0$, when $x = 0$; $\therefore$ $C' = 0$. If $x = l$, $v = 0$, and $P_1 = (\frac{1}{24} - \frac{1}{6})wl^4 \div (\frac{1}{6} - \frac{1}{2})\,l^3 = \frac{3}{8}wl.$ $F_x = \frac{3}{8}wl - wx.$

Substitute this value of P_1 in the above equations.

$$\frac{d^2v}{dx^2} = \frac{1}{\mathrm{E\,I}}\left(\tfrac{3}{8}wlx - \tfrac{1}{2}wx^2\right) = \frac{w}{2\mathrm{E\,I}}\left(\tfrac{3}{4}lx - x^2\right);$$

$$\frac{dv}{dx} = \frac{w}{2\mathrm{E\,I}}\left(\tfrac{3}{8}lx^2 - \tfrac{1}{3}x^3 - \tfrac{1}{24}l^3\right);\quad v = \frac{w}{8\mathrm{E\,I}}\left(\tfrac{1}{2}lx^3 - \tfrac{1}{3}x^4 - \tfrac{1}{6}l^3x\right).$$

$$\text{For } v \text{ max., make } \frac{dv}{dx} = 0, \text{ or } \tfrac{3}{8}lx^2 - \tfrac{1}{3}x^3 = \tfrac{1}{24}l^3.$$

$$\therefore \quad x^3 - \tfrac{9}{8}lx^2 = -\tfrac{1}{8}l^3.$$

As a minimum value of v, or $v = 0$, occurs for $x = l$, divide $8x^3 - 9lx^2 + l^3 = 0$ by $x - l = 0$, obtaining

$$8x^2 - lx - l^2 = 0, \text{ or } x = l\,\frac{1 \pm \sqrt{33}}{16} = 0.4215l.$$

$$\text{Then } v \text{ max.} = -0.0054\,\frac{(wl)l^3}{EI}.$$

To find points of M max. put $F_x = 0$, or $x = \frac{3}{8}l$.

Also, by inspection, M max. when $x = l$.

For $x = \frac{3}{8}l$, M max. $= (\frac{9}{64} - \frac{9}{128})wl^2 = \frac{9}{128}wl^2$.

For $x = l$, M max. $= (\frac{3}{8} - \frac{1}{2})\,wl^2 = -\frac{1}{8}wl^2$

For the point of contraflexure, $\frac{3}{8}lx - \frac{1}{2}x^2 = 0$; or $x = \frac{3}{4}l$; as was to be expected from the position of the point of maximum positive M.

Note again that the point of maximum bending moment is *not* the point of maximum deflection.

It will be seen that a continuous beam of two equal spans l, uniformly loaded with w per unit, has end reactions of $\frac{3}{8}wl$, and a central reaction of $2 \times \frac{5}{8}wl = \frac{5}{4}wl$; that points of contraflexure divide each span at $\frac{1}{4}l$ from the middle pier; and that the bending moment at the middle of the remaining segment of $\frac{3}{4}l$ is, as above, $\frac{3}{4} \cdot \frac{3}{4} \quad \frac{1}{8}wl^2 = \frac{9}{128}wl^2$. It will also be seen that, since the bending moment at P_2 is $-\frac{1}{8}wl^2$, a uniform beam, continuous over two equal spans, each l, is no stronger than the same beam of span l with the same uniform load. It is, however, about two and a half times as stiff.

Example.—A girder spanning two equal openings of 15 ft., and carrying a 16 in. brick wall 10 ft. high, of 110 lbs. per cubic ft., will throw a load of $\dfrac{5 \cdot 4 \cdot 110 \cdot 10 \cdot 15}{4 \cdot 3} = 27{,}500$ lbs. on the middle post, and must resist a bending moment of

$$\frac{4 \cdot 110 \cdot 10 \cdot 15 \cdot 15}{3 \cdot 8} = 41{,}250 \text{ ft. lbs.}$$

120. Two-Span Beam, with Middle Support Lowered.—A uniform beam, uniformly loaded, and supported at its ends, will have a certain deflection at the middle which can be calculated. If the middle point is then lifted by a jack, until returned

to the straight line through the two end supports, the pressure on the jack, by § 119, will be five-eighths of the load on the beam. Since deflection is proportional to the weight, other things being equal,—if the jack is then lowered one-fifth of the first deflection referred to, the pressure on the jack will be reduced one-fifth, or to one-half of the load on the beam. Hence, if a uniformly loaded beam of two equal, continuous spans has its middle support lower than those at its ends by one-fifth of the above deflection, the middle reaction will be one-half the whole weight, the *bending moment* will be *zero* at the middle, and the beam may be cut at that point without disturbance of the forces.

121. **Beam of Span l,** fixed at left and supported at right end, and carrying a single weight W at a distance a from the fixed end, Fig, 55. Origin at fixed end.

$$M_o = -\frac{W}{2l^2} a(l-a)(2l-a);$$

$$M_a = \frac{Wa}{2l^3} a(l-a)(3l-a).$$

$$F_o = \frac{W}{2l^3} a(l-a)(2l-a);$$

$$P_2 = \frac{Wa}{2l^3} a(3l-a).$$

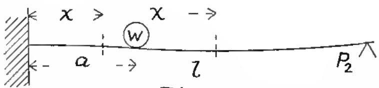
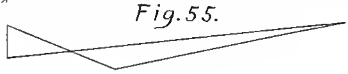


Fig. 55.



The reaction at the supported end, being at present unknown, will be denoted by P_2, and moments will be taken on the right of any section x. From lack of symmetry, separate expressions must be written for segments on either side of W.

BETWEEN W AND FIXED END.

$$M_x = P_2(l-x) - W(a-x)$$

$$\frac{dv}{dx} = A[P_2(lx - \tfrac{1}{2}x^2) - W(ax - \tfrac{1}{2}x^2) + C]$$

$$v = A\left[P_2\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) - W\left(\frac{ax^2}{2} - \frac{x^3}{6}\right) + Cx + C'\right]$$

$\frac{dv}{dx} = 0$, when $x = 0$; ∴ $C = 0$.

$v = 0$ when $x = 0$; ∴ $C' = 0$.

BETWEEN W AND SUPPORTED END.

$$M_x = P_2(l-x) \qquad \frac{1}{EI} = A.$$

$$\frac{dv}{dx} = A[P_2(lx - \tfrac{1}{2}x^2) + C'']$$

$$v = A\left[P_2\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C''x + C'''\right]$$

If $x = a$, $\frac{dv}{dx}$ on left $= \frac{dv}{dx}$ on right.

If $x = a$, v on left $= v$ on right.

$$\therefore \quad C'' = W(a^2 - \tfrac{1}{2}a^2) = -\tfrac{1}{2}Wa^2; \; C''' = W(-\tfrac{1}{2}a^3 + \tfrac{1}{6}a^3 + \tfrac{1}{2}a^3) = \tfrac{1}{6}Wa^3.$$

If $x = l$, v at $P_2 = 0$; ∴ $P_2(\tfrac{1}{2}l^3 - \tfrac{1}{6}l^3) - \tfrac{1}{2}Wa^2l + \tfrac{1}{6}Wa^3 = 0$,

$$\text{or } P_2 = Wa^2(\tfrac{1}{2}l - \tfrac{1}{6}a) \div \tfrac{1}{3}l^3 = \frac{Wa^2}{2l^3}(3l-a).$$

If this value of P_2 is substituted in the above equations the desired expressions are obtained. Thus

$$M_x = \frac{Wa^2}{2l^3}(3l - a)(l - x) - W(a - x) \quad \Big| \quad M_x = \frac{Wa^2}{2l^3}(3l - a)(l - x).$$

M max., by inspection, when $x = 0$, or $x = a$.

M max. $= \frac{Wa^2}{2l^2}(3l - a) - Wa = -\frac{W}{2l^2}a(l - a)(2l - a)$, at fixed end; and $= \frac{Wa^2}{2l^3}(l - a)(3l - a)$ at the weight.

The point of contraflexure occurs between W and the fixed end, where $M = 0$, or

$$\frac{a^2}{2l^3}(3l - a)(l - x) - (a - x) = 0. \quad \therefore \quad x = al\frac{2l - a}{2l(l + a) - a^2}.$$

The maximum deflection will be found where $\frac{dv}{dx} = 0$, on on the right or left, according to the value of a.

Various problems may be devised, such as those for finding values of a which will make $\frac{d^2v}{dx^2}$, $\frac{dv}{dx}$ or v a maximum, with the position of corresponding points of contraflexure and maximum deflection. They are more curious than useful.

In solving the more intricate problems in the flexure of beams, as well as those just treated, each equation of condition can be used but once in the same problem, and as many unknown quantities can be determined as there are independent equations of condition. The reactions and moments at the points of support are usually unknown, and must be found by the aid of such flexure equations as have just been used.

The above beam may be regarded in the light of two equal continuous spans with W on each, distant a each side of the middle point of support.

Example.—A bridge stringer which is continuous over two successive openings of 12 ft. each, and carries a weight from the wheels of a wagon, of 3,000 lbs. at each side of and 3 ft. from the middle support, will be horizontal over that support. Then $-$M max. $= -\frac{3,000}{2 \cdot 12^2} \cdot 3 \cdot 9 \cdot 21 = -5,906.25$ ft. lbs. $+$M max. $= \frac{3,000}{2 \cdot 12^3} 3^2 \cdot 33 \cdot 9 = 2,320.3$ ft. lbs. $P_2 = \frac{3,000}{2 \cdot 12^3} \cdot 3^2 \cdot 33 = 258$ lbs. Reaction at middle support from both spans $= 2(3,000 - 258) = 5,484$ lbs.

CONTINUOUS BEAMS.

122. **Clapeyron's Formula, or the Three Moment Theorem for Continuous Loading.**—To find the reactions, shears and bending moments for a horizontal, uniform, continuous beam, loaded with w_1, w_2, w_3, etc., loads per running unit over the successive spans l_1, l_2, l_3, etc. Fig. 56. P_0, P_1, P_2, etc., denote the unknown reactions; M_0, M_1, M_2, etc., the unknown bending moments at points of support, A, B, C, etc.; F_0, F_1, F_2, etc., denote the shears immediately to the right of A, B, C, etc.; while F'_1, F'_2, etc., denote the shears immediately to the left of the points of support B, C, etc.

The origin of co-ordinates is first taken at A, and the supports are on a level. $+$ M makes the beam concave on the upper side. As positive shear acts upward at the left of any cross-section, w is negative.

Consider the condition of equilibrium of the first span A B, or l_1, loaded throughout with w_1 per unit of length.

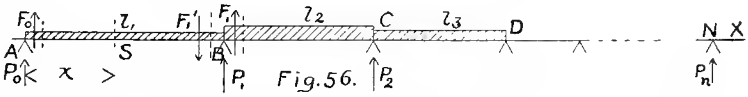


Fig. 56.

Take moments on the left side of and about a section S, distant x from the origin A. The bending moment at S will be, by § 68,

$$M = E\,I\,\frac{d^2 v}{dx^2} = M_0 + F_0\,x - \tfrac{1}{2}w_1 x^2. \qquad (1.)$$

Let i_0, i_1, i_2 . . . i_n = tangent of inclination of the neutral axis at A, B, C. . . N. Integrate (1.) between the limits o and x, transposing the slope i_0 for the limit zero to the left hand member, and thus obtain an expression for the difference in slope or inclination of the two tangents to the bent beam at A and S.

$$E\,I\left(\frac{dv}{dx} - i_0\right) = M_0\,x + \tfrac{1}{2}F_0\,x^2 - \tfrac{1}{6}w_1 x^3. \qquad (2.)$$

When $x = l_1$, $\dfrac{dv}{dx} = i_1$, and hence

$$E\,I\,(i_1 - i_0) = M_0\,l_1 + \tfrac{1}{2}F_0\,l_1^2 - \tfrac{1}{6}w_1 l_1^3. \qquad (3.)$$

Integrate (2.) and determine constant as zero, because $v = 0$, when $x = 0$.

$$E\,I\,(v - i_0\,x) = \tfrac{1}{2}M_0\,x^2 + \tfrac{1}{6}F_0\,x^3 - \tfrac{1}{24}w_1 x^4. \qquad (4.)$$

Make $x = l_1$; then $v = v_1 = 0$, and

$$-E\,I\,i_0\,l_1 = \tfrac{1}{2}M_0\,l_1^2 + \tfrac{1}{6}F_0\,l_1^3 - \tfrac{1}{24}w_1 l_1^4, \text{ or}$$

$$-E\,I\,i_0 = \tfrac{1}{2}M_0\,l_1 + \tfrac{1}{6}F_0\,l_1^2 - \tfrac{1}{24}w_1 l_1^3. \qquad (5.)$$

Eliminate i_0 by subtracting (5.) from (3.).

$$E\,I\,i_1 = \tfrac{1}{2}M_0\,l_1 + \tfrac{1}{3}F_0\,l_1^2 - \tfrac{1}{8}w_1 l_1^3. \qquad (6.)$$

If the origin is taken at B instead of A, an equation like (5.) is obtained for the second span l_2, or

$$-E\,I\,i_1 = \tfrac{1}{2}\,M_1\,l_2 + \tfrac{1}{6}\,F_1\,l_2^2 - \tfrac{1}{24}\,w_2\,l_2^3. \qquad (7.)$$

Add (6.) and (7.), obtaining

$$0 = \tfrac{1}{2}\,M_0 l_1 + \tfrac{1}{2}\,M_1\,l_2 + \tfrac{1}{3}\,F_0\,l_1^2 + \tfrac{1}{6}\,F_1\,l_2^2 - \tfrac{1}{8}\,w_1\,l_1^3 - \tfrac{1}{24}\,w_2\,l_2^3. \qquad (8.)$$

The unknown slopes have thus been eliminated. The next step is to remove either M or F. Equation (1.) must equal M_1 for $x = l_1$; therefore

$$M_1 = M_0 + F_0\,l_1 - \tfrac{1}{2}\,w_1\,l_1^2, \text{ or } F_0 = \frac{M_1 - M_0}{l_1} + \tfrac{1}{2}\,w_1\,l_1.$$

In the same way, for 2d. span, $F_1 = \dfrac{M_2 - M_1}{l_2} + \tfrac{1}{2}\,w_2\,l_2$.

Substitute the values in (8.) and obtain

$$0 = \frac{M_0\,l_1}{2} + \frac{M_1\,l_2}{2} + \frac{M_1 - M_0}{3}\,l_1 + \frac{w_1\,l_1^3}{6} + \frac{M_2 - M_1}{6}\,l_2 + \frac{w_2\,l_2^3}{12}$$
$$- \frac{w_1\,l_1^3}{8} - \frac{w_2\,l_2^3}{24}. \quad \therefore$$

$$M_0\,l_1 + 2\,M_1\,(l_1 + l_2) + M_2\,l_2 = -\tfrac{1}{4}\,(w_1\,l_1^3 + w_2\,l_2^3). \qquad (9.)$$

which is Clapeyron's formula for pier moments for a continuous beam, with continuous load, uniform per span. Notice the symmetry of the expression. The negative sign to the second member indicates that the bending moments at points of support are usually negative.

Example.—Three spans, 30 ft., 60 ft. and 30 ft. in succession. Load on first and last 500 lbs. per ft., on middle span 300 lbs. per ft. No moment at either outer end. Then $M_0 = 0$. $M_1 = M_2$ by symmetry. $2M_1 \,.\, 90 = -\tfrac{1}{4}\,(500 \,.\, 30^3 + 300 \,.\, 60^3)$. $M_1 = -108{,}750$ ft. lbs. $F_0 = -3{,}625 + 7{,}500 = +3{,}875$ lbs. $F'_1 = +3{,}875 - 30 \,.\, 500 = -11{,}125$ lbs. $F_1 = 300 \,.\, 30 = 9{,}000$

lbs. $\therefore P_0 = P_4 = 3,875$ lbs; $P_1 = P_2 = 11,125 + 9,000 = 20,125$ lbs. The bending moment and shear at any point can now be readily determined.

If the two adjacent spans are equal and have the same load, $M_0 + 4 M_1 + M_2 = -\frac{1}{2} wl^2$. (10.)

If there are n spans, $n - 1$ equations can be written between $n + 1$ quantities M_0, $M_1 \ldots M_n$. But it the beam is simply placed on the points of support, the extremities being unrestrained, $M_0 = 0$ and $M_n = 0$, and there remain $n - 1$ equations to determine $M_1 \ldots M_{n-1}$. If the beam is fixed at the ends, the equations $i_0 = 0$ and $i_n = 0$ will complete the required number.

123. **Shears and Reactions.**—As the shear is the first derivative of the bending moment, § 68, from (1.) is obtained

$$\frac{d\,M'}{dx} = F = F_0 - w_1 x, \qquad (11.)$$

as was to be expected, $+ F$ acting upwards on the left of the section. A similar equation can be written for each span.

The reaction at any point of support will be equal to the shear on its right plus that on its left with the sign reversed. As the shear on its left is usually negative, the arithmetical sum of F_n and F'_n commonly gives the reaction.

A simple example may make the application plainer. Given two equal spans, on three supports.

$$w_1 = w_2 = w. \quad M_0 = 0,\ M_2 = 0. \quad (10.) \text{ gives } M_1 = -\tfrac{1}{8} wl^2$$
$$F_0 = -\tfrac{1}{8} wl + \tfrac{1}{2} wl = \tfrac{3}{8} wl;\ F'_1 = \tfrac{3}{8} wl - wl = -\tfrac{5}{8} wl.$$
$$F_1 = \tfrac{1}{8} wl + \tfrac{1}{2} wl = \tfrac{5}{8} wl;\ F'_2 = \tfrac{5}{8} wl - wl = -\tfrac{3}{8} wl.$$
$$P_0 = \tfrac{3}{8} wl;\ P_1 = (\tfrac{5}{8} + \tfrac{5}{8})\, wl = \tfrac{5}{4} wl;\ P_2 = \tfrac{3}{8} wl.$$

$$(5.) \text{ gives } i_0 = -\frac{1}{E\,I}\left(0 + \tfrac{3}{48} - \tfrac{1}{24}\right) wl^3 = -\frac{wl^3}{48 E\,I}.$$

$$(6.) \text{ gives } i_1 = \frac{1}{E\,I}\left(0 + \tfrac{3}{24} - \tfrac{1}{8}\right) wl^3 = 0$$

and the analogous equation for the second span is

$$i_2 = \frac{1}{E\,I}\left(-\tfrac{1}{16} + \tfrac{5}{24} - \tfrac{1}{8}\right) wl^3 = \frac{wl^3}{48 E\,I},$$

which differs from i_0 only in direction of slope.

$$(2.) \text{ gives } E\,I\,\frac{dv}{dx} = -\frac{wl^3}{48} + \tfrac{3}{16} wlx^2 - \frac{wx^3}{6}.$$

$$(4.) \text{ gives } E\,I\,v = -\frac{wl^3}{48} x + \tfrac{3}{48} wlx^3 - \frac{wx^4}{24}.$$

These equations determine the slope and deflection at each point. Putting $\frac{dv}{dx} = 0$, there results $l^3 - 9\,lx^2 + 8\,x^3 = 0$, containing the root $x = l$, already known. Therefore divide by $l - x = 0$, and obtain $l^2 + lx - 8\,x^2 = 0$, which is satisfied for $x = 0.4215\,l$, the point of max. deflection. The substitution of this value in the equation for v will yield v max.

$$\text{From (1.) } M = \tfrac{3}{8}\,wlx - \tfrac{1}{2}\,wx^2.$$

If $M = 0$, $\frac{3}{8}\,l - \frac{1}{2}\,x = 0$, or $x = \frac{3}{4}\,l$, the point of contraflexure.

Differentiate M, and get $F = \frac{3}{8}\,wl - wx$.

If $F = 0$, $x = \frac{3}{8}\,l$, the point of $+$ M max. $\therefore$

$$M \text{ max.} = \tfrac{3}{8}\,wl \,.\, \tfrac{3}{8}\,l - \tfrac{1}{2}\,w \,.\, \tfrac{9}{64}\,l = \tfrac{9}{128}\,wl^2$$

Example.—If a uniformly loaded continuous beam covers five equal spans,

$$M_0 + 4\,M_1 + M_2 = -\tfrac{1}{2}\,wl_2 = M_1 + 4\,M_2 + M_3$$
$$= M_2 + 4\,M_3 + M_4 = M_3 + 4\,M_4 + M_5.$$

$$M_0 = 0;\ M_5 = 0. \quad \text{Then } M_1 = -\tfrac{4}{38}\,wl^2 = M_4;$$
$$M_2 = -\tfrac{3}{38}\,wl^2 = M_3.$$

$$F_0 = \tfrac{15}{38}\,wl;\ F_1' = -\tfrac{23}{38}\,wl;\ F_1 = \tfrac{20}{38}\,wl;\ F_2' = -\tfrac{18}{38}\,wl;$$
$$F_2 = \tfrac{19}{38}\,wl, \text{ etc.}$$

$$P_0 = \tfrac{15}{38}\,wl = P_5;\ P_1 = \tfrac{43}{38}\,wl = P_4;\ P_2 = \tfrac{37}{38}\,wl = P_3.$$

The sum of the reactions must equal 5 wl.

124. Coefficients for Moments and Shears.—It has been found that the numerical coefficients for moments and shears at the points of support, when all spans are equal and the load is uniform throughout, may be tabulated easily for reference and use. Thus the values of M and F just obtained for the five equal spans can be selected from the lines marked V. The reactions are given by the *arithmetical* addition of the shears. The sum of the reactions must equal the total load. The shears at the two ends of any span differ by the whole load on the span, the shear at the right end being negative. The dashes represent the spans.

SHEAR AND REACTION COEFFICIENTS.

I. $\frac{1}{2} — \frac{1}{2}\,wl$

II. $\frac{3}{8} — \frac{5}{8}\ \frac{5}{8} — \frac{3}{8}\,wl$

III. $\frac{4}{10} — \frac{6}{10}\ \frac{5}{10} — \frac{5}{10}\ \frac{6}{10} — \frac{4}{10}$

IV. $\frac{11}{28} — \frac{17}{28}\ \frac{15}{28} — \frac{13}{28}\ \frac{13}{28} — \frac{15}{28}\ \frac{17}{28} — \frac{11}{28}$

V. $\frac{15}{38} — \frac{23}{38}\ \frac{20}{38} — \frac{18}{38}\ \frac{19}{38} — \frac{19}{38}\ \frac{18}{38} — \frac{20}{38}\ \frac{23}{38} — \frac{15}{38}$

etc., etc.

PIER MOMENT COEFFICIENTS.

II. $-\frac{1}{8}-wl^2$.

III. $-\frac{1}{10}-\frac{1}{10}-wl^2$.

IV. $-\frac{3}{28}-\frac{2}{28}-\frac{3}{28}-$

V. $-\frac{4}{38}-\frac{3}{38}-\frac{3}{38}-\frac{4}{38}-$

etc., etc.

The rule for writing either table is as follows: For an even number of spans, the numbers in any horizontal line are obtained by multiplying the fraction above, in any *diagonal* row, both numerator and denominator, by *two*, and adding the numerator and denominator of the preceding fraction. Thus, in the first table, $\frac{2 \times 6+5}{2 \times 10+8}=\frac{17}{28}$, and in the second table, $\frac{2 \times 1+1}{2 \times 10+8}=\frac{3}{28}$, or $\frac{2 \times 3+2}{2 \times 38+28}=\frac{8}{104}$. For an odd number of spans, add the two preceding fractions in the same diagonal row, numerator to numerator and denominator to denominator. Thus, $\frac{13+5}{28+10}=\frac{18}{38}$. The denominators agree in both tables. A recollection of two or three quantities will enable one to write all the others.

Example.—Continuous beam of 5 equal spans, each l, carrying w per ft. Where and what is the max. $+$ M in second span. Shear changes sign at $\frac{20}{38}\,l$ from left end of span. If this span were independent, $+$ M at that point would be $\frac{1}{8}\,wl^2 \cdot \frac{20 \cdot 18}{19 \quad 19}$ $=\frac{360}{361}\cdot\frac{wl^2}{8}$. The negative or subtractive moment is $(\frac{3}{38}+\frac{1}{38}\cdot\frac{18}{38})\,wl^2$ The difference between these values is $+$ M max.

A more general investigation will produce equations which are of great practical value in the solution of problems concerning continuous bridges, swing bridges, etc., as follows:—

125. Three-Moment Theorem for a Single Weight.—O is the origin, Fig. 57; the supports are at distances h below the axis of x. A single weight W_n is distant kl_n from O on the span l_n, k being a fraction, less than unity, of the span in which W is situated.

The moment at section S beyond W_n will be, as in the former discussion,

$$M_x = M_n + F_n x - W_n(x - kl_n). \qquad (1.)$$

If $x = l_n$, $M_x = M_{n+1}$, and from (1.)

$$F_n = \frac{M_{n+1} - M_n}{l_n} + W_n(1 - k). \qquad (1a.)$$

For an unloaded span, $W = 0$, and $F_m = \dfrac{M_{m+1} - M_m}{l_m}$.

For the shear on the left of a section at the right end of the nth span,

$$F'_{n+1} = F_n - W_n = \frac{M_{n+1} - M_n}{l_n} - W_n k.$$

For an unloaded span, $W = 0$, and $F'_m = \dfrac{M_m - M_{m-1}}{l_{m-1}}$.

As F'_m is the shear at left of support m, and F_m is the shear at right of the same support, the reaction there will be the sum of F_m and $-F'_m$ or

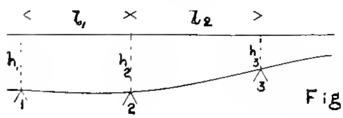
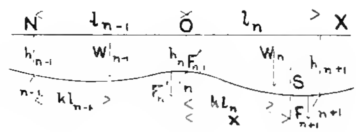


Fig. 57.



$$P_m = \frac{M_{m+1} - M_m}{l_m} + \frac{M_{m-1} - M_m}{l_{m-1}}.$$

$$\int_0^x E\,I \frac{d^2v}{dx^2}\,dx = \int_0^x M_x\,dx = M_n \int_0^x dx + F_n \int_0^x x\,dx$$
$$- W_n \int_{kl_n}^x (x - kl_n)\,dx. \qquad (2.)$$

Note that the integral of the last term is between limits kl_n and x only.

$$E\,I\left(\frac{dv}{dx} - i_n\right) = M_n x + \tfrac{1}{2}F_n x^2 - \tfrac{1}{2}W_n (x - kl_n)^2. \qquad (3.)$$

Since the origin is at a distance h_n above the nth support, the constant for the next integration is h_n.

$$E\,I(v - i_n x - h_n) = \tfrac{1}{2}M_n x^2 + \tfrac{1}{6}F_n x^3 - \tfrac{1}{6}W_n (x - kl_n)^3, \qquad (4.)$$

which is the general equation of the curve of the neutral axis, the term in W disappearing for values of x less than kl_n.

If $x = l_n$, $v = h_{n+1}$. If the value of F_n from $(1a)$ is inserted in (4.), the slope at support n is

$$i_n = \frac{h_{n+1} - h_n}{l_n} - \frac{1}{6E\,I}[2M_n l_n + M_{n+1} l_n$$
$$+ W_n l^2_n (2k - 3k^2 + k^3)]. \qquad (5.)$$

The equation of the curve is therefore completely determined when M_n and M_{n+1} are known. The equation of this curve, between W_n and the $n+1$th. support, is given by (4.), and the tangent of its angle with the axis of x by (3.). If the value of F_n from (1*a*.) and of i_n from (5.) are substituted in (3.); and $x = l_n$, $\frac{dv}{dx}$ will be the tangent i_{n+1} at $n+1$, or

$$i_{n+1} = \frac{h_{n+1} - h_n}{l_n} + \frac{1}{6EI}[M_n l_n + 2M_{n+1} l_n + W_n l_n^2 (k - k^3)].$$

Remove the origin from O to N, and derive an expression for i_n by diminishing the indices.

$$i_n = \frac{h_n - h_{n-1}}{l_{n-1}} + \frac{1}{6EI}[M_{n-1} l_{n-1} + 2M_n l_{n-1} + W_{n-1} l_{n-1}^2 (k - k^3)].$$

Equate with (5.) and transpose.

$$M_{n-1} l_{n-1} + 2M_n (l_{n-1} + l_n) + M_{n+1} l_n$$
$$= 6EI\left(\frac{h_{n-1} - h_n}{l_{n-1}} + \frac{h_{n+1} - h_n}{l_n}\right) - W_{n-1} l_{n-1}^2 (k - k^3)$$
$$- W_n l_n^2 (2k - 3k^2 + k^3),$$

which is the most general form of the Three Moment Theorem for a girder of constant cross-section.

$k - k^3 = k(1 - k)(1 + k)$; $2k - 3k^2 + k^3 = k(1 - k)(2 - k)$.

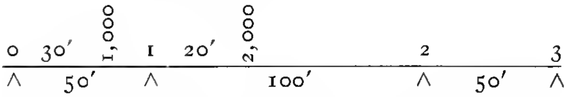
Pier moments are usually negative and the end moments zero. When the supports are on a level $h_1 = h_2$ etc., and the term in E I disappears.

Any reaction $P_n = F_n - F'_n$;

$$P_n = \frac{M_{n+1} - M_n}{l_n} + \frac{M_{n-1} - M_n}{l_{n-1}} + W_n (1 - k) + W_{n-1} k.$$

See also Greene's Graphics, Part II., Bridge Trusses, Chap. VIII.

Example.—Three span continuous girder as shown, carrying 1,000 lbs. in first span and 2,000 lbs. in second span, at points indicated. Supports on a level.



$$300\ M_1 + 100\ M_2 = -1{,}000 \cdot 2{,}500 \cdot \tfrac{6}{10} \cdot \tfrac{64}{100} - 2{,}000 \cdot 10{,}000 \cdot \tfrac{2}{10} \cdot \tfrac{8}{10} \cdot \tfrac{18}{10}$$

$$100\ M_1 + 300\ M_2 = -2{,}000 \quad 10{,}000 \cdot \tfrac{2}{10} \cdot \tfrac{96}{100}.$$

$$\tfrac{1}{2}(M_1 + M_2) = -13{,}200, \quad \tfrac{1}{2}(M_1 - M_2) = -7{,}200,$$

$$M_1 = -20{,}400 \text{ ft. lbs.} \qquad M_2 = -6{,}000 \text{ ft. lbs.}$$

$$P_0 = \frac{-20{,}400}{50} + \frac{1{,}000 \cdot 4}{10} = -8 \text{ lbs.} \quad P_1 = \frac{-6{,}000 + 20{,}400}{100}$$

$$+ \frac{20{,}400}{50} + \frac{2{,}000 \cdot 8}{10} + \frac{1{,}000 \cdot 6}{10} = 2{,}752 \text{ lbs.}$$

$$P_2 = \frac{6{,}000}{50} + \frac{-20{,}400 + 6{,}000}{100} + \frac{2{,}000 \cdot 2}{10} = 376 \text{ lbs.}$$

$$P_3 = \frac{-6{,}000}{50} = -120 \text{ lbs.}$$

Examples.—1. A brick wall 16 in. thick, 12ft. high, and 32 ft long, weighing 108 lbs. per cubic ft., is carried on a beam supported by four columns, one at each end, and one 8 ft. from each end. Find M at the two middle columns, the reactions, and the value of I, if $f = 16{,}000$ lbs.

$M = -31{,}104$ ft. lbs.; $P_1 = 3{,}024$ lbs.; $P_2 = 24{,}624$ lbs.

2. Two successive openings of 8 ft. each, are to be spanned. Which will be stronger for a uniform load, two 8 ft. joists end to end, or one 16 ft. long? Find their relative stiffness.

3. A beam of three equal spans carries a single weight. What will be the reactions and their signs at the third and fourth points of support, when W is in the middle of the first span?

$-\tfrac{3}{20}W; + \tfrac{1}{40}W.$

4. A beam loaded with 50 lbs. per foot rests on two supports 15 ft. apart and projects 5 ft. beyond at one end. What additional weight must be applied to that end to make the beam horizontal at the nearer point of support?

$156\tfrac{1}{4}$ lbs. at the end, or $312\tfrac{1}{2}$ lbs. distributed.

CHAPTER VIII.

PIECES UNDER TENSION.

126. **Central Pull.**—If the resultant tension P acts along the axis of the piece, the stress may be considered as uniformly distributed on the cross-section S. If, then, f is the maximum safe working stress per square inch for the kind of load which causes P, Fig. 58,

$$P = fS; \text{ or } S = P \div f,$$

for the necessary section, which need not be exceeded throughout that portion of the piece where the above conditions apply. Changes due to connections will require a larger section.

If, owing to lack of uniformity in the material, or the direct application of P, at the end of a wide bar, to a limited portion only of the width, the stress may not be considered as uniformly distributed at a particular cross-section, injurious stress may be prevented by taking the mean stress f at a smaller value, and obtaining a larger cross-section.

If there is lack of homogeneity, or two materials are used together, or two or more bars work side by side, those fibres which offer the greatest resistance to stretching will be subject to the greatest stress. Fortunately, the slight yielding and bending of connecting parts tend to restore equality of action.

A long tension member has a much greater resisting power against a suddenly applied load than a short one of equal strength per square inch.

127. **Eccentric Pull.**—If the variation of stress on a cross-section is due to the fact that the line of action of the applied force does not traverse the centre of figure of the cross-section S, the force P′ that can be imposed without causing a unit stress greater than f at any point in the section is less than P of the preceding formula, and depends upon the

perpendicular distance y_0 of the action line of P′ from the centre of S.

For safe stresses, which must lie well within the elastic limit, the unit stress is proportional to the stretch, and plane cross-sections of the bar before the force is applied are assumed to remain plane after the bar is stretched. It is impossible to detect experimentally that this assumption is not true. Were the plane sections to become even slightly warped, the cumulative warpings of successive sections in a long bar ought to become apparent to the eye. No reference is intended here to local distortion preceding failure.

If the stress on any section is not uniform and the successive sections remain plane, they must be a little inclined to one another. The stress on any cross-section S must therefore vary uniformly in the direction of the deviation of the action line of P′ from the centre, Fig. 58, and be constant on lines at right angles to that deviation. The stress on each particle may be divided into two parts, the mean stress, which is always the existing stress at the centre of the section, and the variable part. The mean stress balances the force P′; but the moment of P′ about the centre, or P′y_0, must be balanced by the moment of the variable part of the stress, taken about the axis in the plane of and through the centre of the cross-section, perpendicular to y_0. Take the origin at the centre.

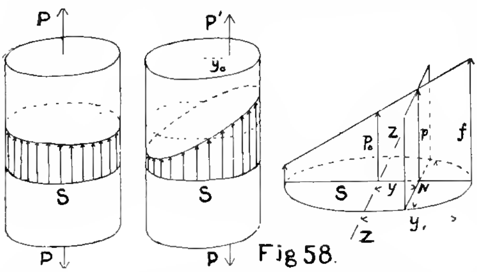


Fig 58.

Let p = unit stress at the point distant y from the centre, measured in the direction y_0 and y_1, which latter is the distance to the edge where the unit stress is f. If p_0 = mean unit stress found at the centre, $p - p_0$ will represent the variable part found at the distance y.

$$p - p_0 \cdot f - p_0 = y : y_1, \text{ or } p - p_0 = \frac{f - p_0}{y_1} y.$$

If z = variable width of section at distance y from the centre, the moment of the variable portion of the stress about the axis $Z\,Z$ through the centre will be

$$M = \int_{-y_1}^{y_1} \frac{f - p_0}{y_1}\, y \,.\, z dy \,.\, y = \frac{f - p_0}{y_1} \int_{-y_1}^{y_1} y^2 z dy = \frac{f - p_0}{y_1}\, I_z\,,$$

where I_z denotes the moment of inertia of the cross-sectional area about the axis Z.

This expression is the resisting moment of the cross-section. But $M = P'y_0$, and $p_0 = P' \div S$. $\therefore$

$$P'y_0 = \left(f - \frac{P'}{S}\right)\frac{I_z}{y_1}; \quad \text{or } P' = \frac{fS}{1 + \dfrac{y_0 y_1 S}{I}} = \frac{fS}{1 + \dfrac{y_0 y_1}{r^2}},$$

where $r^2 = I_z \div S$, to be measured in the direction y_1.

Also $f = \dfrac{P'}{S}\left(1 + \dfrac{y_0 y_1}{r^2}\right)$, which gives the max. unit stress due to P' and y_0.

Example.—A square bar, 1 in. in section, carries 6,000 lbs. tension. The centre of the eye at the end is $\frac{1}{4}$ in. out of line. Then $f = 6{,}000(1 + \frac{1}{4} \cdot \frac{1}{2} \quad 12) = 15{,}000$ lbs. per sq. in., $2\frac{1}{2}$ times the mean and probably the intended stress.

A bar which is not perfectly straight before tension is applied to it, tends to straighten itself under a pull, but the stress will not become uniform on a cross-section. The bar is weaker in the ratio of p_0 to f, as it might carry fS if the force were central, but now can safely carry only $p_0 S$. If a thrust is applied to a bent bar, there is a tendency to increased deviation from a straight line, and to an increase in the variation of stress.

It is seen from the example above, that a small deviation y_0 will have a decided effect in increasing f for a given P', or in diminishing the allowable load for a given unit stress. Herein may be the explanation of some considerable variations of the strength of apparently similar pieces under test; and, on account of such effect, added to other reasons, allowable working stresses may well be and are reduced below what otherwise might be used.

128. Hooks.—The bending action on and the strength of a hook are given by the same formulas. Here y_1 will be the distance from the *inside* edge to the centre of the cross-section, and y_0 the distance from the action line of the load to the same centre. Then the max. unit tension

$$f = \frac{P}{S}\left(1 + \frac{y_0 y_1}{r^2}\right).$$

Example.—A hook, the section of which in the bend is elliptical, 1 in. $\times$ $\frac{5}{8}$ in., carries the link of a chain at a distance of $\frac{1}{2}$ in. horizontally from the inside of the bend. Then $S = \frac{22}{7} \cdot \frac{1 \cdot 5}{8 \cdot 4} = \frac{1}{2}$ sq. in.; $r^2 = \frac{h^2}{16} = \frac{1}{16}$, § 94, V.; $y_0 = 1$ in. Then $f = 2P(1 + 1 \cdot \frac{1}{2} \cdot 16) = 18P$. If $f = 8{,}000$, $P = 450$ lbs.; if $f = 12{,}000$, $P = 650$ lbs. Compare with the given section. The ordinate of the bend should be reduced as much as possible.

129. Combined Tie and Beam.—If to a tension member transverse forces are applied, or if it is horizontal and its weight is of importance, the unit tensile stress on the convex edge, due to the maximum bending moment, must be added to the unit stress at that point due to the direct pull. The former,

$$f' = \frac{(M \text{ max.})y_1}{I}, \text{ and the latter } f'' = \frac{P}{S}.$$

But $f = f' + f''$ must not exceed the safe unit tension, and the needed section is, since $I = Sr^2$,

$$S = \frac{1}{f}\left(\frac{(M \text{ max.})y_1}{r^2} + P\right).$$

In this case the sections may vary, since the external bending moment M varies from point to point.

If the piece is rectangular in section, as with timber, the formula may be written,

$$f = \frac{6M}{bh^2} + \frac{P}{bh}; \text{ or } b = \frac{1}{fh}\left(\frac{6M}{h} + P\right).$$

In practical calculation of such a rectangular section, if h is assumed, it is sufficient to compute the breadth to carry M and add enough breadth to carry P, when the combined section will have exactly f at the edge.

Example.—A rectangular wooden beam of 12 ft. span carries a single weight of 3,000 lbs. at the quarter span, and, as part of a truss, resists a pull of 20,000 lbs. If $f = 1,000$ lbs., what should be the section under the weight? M max. $= \frac{3,000 \cdot 3 \cdot 9 \cdot 12}{12}$ $= 81,000$ in. lbs. $\frac{81,000 \cdot 6}{1,000} = bh^2 = 486$. If $h = 12$, $b = 3.37$. Also $\frac{20,000}{1,000 \cdot 12} = 1.67$. Entire breadth $= 3.37 \times 1.67 = 5.04$. Section $= 5 \times 12$ in. The same result is obtained by the formula

$$b = \frac{1}{12 \cdot 1,000}\left(\frac{6 \cdot 81,000}{12} + 20,000\right).$$

130. Action Line of P Moved towards the Concave Side.—It will be economical, if it can be done, in a member having such compound action, to move the line on which P acts towards the concave side. If there are bending moments of opposite signs at different points of the length, or at the same point at different times, such adjustment cannot be made. If y_0 is made equal to ½ (M max.) $\div$ P, one half of the bending moment will be annulled at the point where M max. exists, and at the point of no bending moment from transverse forces an equal amount of bending moment will be introduced. The unit stresses on the extreme fibres at the two sections will be the same, but reversed one for the other.

Example.—A horizontal bar, 6 in. by 1 in. section, and 15 ft. long, has a tension of 33,750 lbs. It carries 100 lbs. per ft. uniformly distributed. M max. $= \frac{100 \cdot 15 \cdot 15 \cdot 12}{8} = 33,750$ in. lbs. $\therefore y_0$ may be made $\frac{1}{2}$ in. Then f from M max. $= \frac{33,750 \cdot 6}{36} =$ $\pm\, 5,625$ lbs. on either edge. But f from P $= \frac{33,750}{6}\left(1 \pm \frac{3}{2} \cdot \frac{12}{6^2}\right)$ $= 5,625\,(1 \pm \frac{1}{2}) = 5,625 \pm 2,812.5$. Stress at top at ends and at bottom at middle — $8,437\frac{1}{2}$ lbs.; at bottom at ends and at top at middle — $2,812\frac{1}{2}$ lbs.

The extreme fibre stress from bending moment of the load varies as the ordinates to a parabola, that from Py_0 is constant. A rectangle super-imposed on a parabolic segment will show the resultant fibre stress at each section.

131. Connecting Rod.—If a bar oscillates laterally rapidly, as does a connecting rod on an engine, or a parallel rod on locomotive drivers, there will be a force developed due to the acceleration, which force will tend to bend the bar as does a distributed load. Each particle of the bar will exert a lateral force (like centrifugal force) of $\frac{w}{g} \cdot \frac{v^2}{r}$ or $\frac{w}{g}\ \alpha^2 r$, where $\alpha = 2\pi n$; $n =$ the num-

ber of double oscillations of the bar, or number of revolutions, per second, of the crank; r = radius of crank, or amplitude of oscillation at the particular point. Then if w = weight of a foot in length of the bar, and g = 32.2 ft., the bar will suffer a bending moment and a resulting fibre stress due to a load $\frac{w}{g} 4\pi^2 n^2 r$ per unit of length, which stress must be added to the tensile stress due to a pull or to the compressive stress due to a thrust. The radius or amplitude r is constant for the parallel rod but varies uniformly along the connecting rod; w may be constant or vary. An I shaped section is suitable for such cases. Owing to the rapid variations and alternations of stress, the maximum unit stress should be small. Mass is disadvantageous in such rods.

132. Tension and Torsion.—A tension bar may be subjected to torsion when it is adjusted by a nut at the end, or by a turnbuckle. The moment of torsion will give rise to a unit shear at the extreme fibre, for a round rod, of $q = \mathrm{T} \div \frac{1}{2}\pi r_1^3$, by § 89, or at the middle of the side for a square rod of $q = \mathrm{T} \div 0.208h^3$, by § 90, either of which, combined with $f = \mathrm{P} \div \mathrm{S}$, the tensile stress, will give $p_1 = \frac{1}{2}f + \sqrt{(\frac{1}{4}f^2 + q^2)}$. § 93.

Example.—A round bar, 2 in. diameter, to be adjusted to a pull of 10,000 lbs. per sq. in., calls for the application to the turnbuckle of 200 lbs. with an arm of 30 in., one half of which moment may be supposed to affect either half of the rod. If the turnbuckle is near one end, the shorter piece will experience the greater part of the moment. $q = \frac{3{,}000 \,.\, 2 \quad 7}{22 \,.\, 1^3} = 1{,}910$ lbs. The max. unit tension on the outside fibres of the rod will be 5,000 $+ \sqrt{(5{,}000^2 + 1{,}910^2)} = 10{,}350$ lbs.

133. Tension Connections.—If a tension member is spliced, or is connected at its ends to other members by rivets, the splice should be so made, or the rivets should be so distributed across the section as to secure a uniform distribution of stress. An angle iron used in tension should be connected by both flanges, Plate III., if the whole section is considered to be efficient. One or more rivet holes must be deducted in calculating the effective section, depending on the spacing of the rivets. See § 217. If the stress is not uniformly distributed on the cross-section, the required size will be found by §127, $\mathrm{S} = \frac{\mathrm{P}'}{f}\left(1 + \frac{y_0 y_1}{r^2}\right)$.

Transverse bolts and bolt holes are similar to rivets and rivet holes.

Timbers may be spliced by clamps with indents, and by scarfed joints, Plate II., in which cases the net section is much reduced; so that timber, while resisting tension well, is not economical for ties, on account of the great waste by cutting away. However, where the tie serves also as a beam, timber may be very suitable.

134. **Screw Threads and Nuts.**—If a metal tie is secured by screw threads and nuts, the section at the bottom of the thread should be some fifteen per cent. larger than the given tension would require, to allow for the local weakening caused by cutting the threads. Bars are often upset or enlarged at the thread, to give the necessary net section, and thus save the material which would be needed for an increase of diameter throughout the length of the rod.

To avoid stripping the thread, the cylindrical surface, whose area is the circumference at the bottom of the thread multiplied by the effective thickness of the nut, should be, when multiplied by the safe unit shear, at least equal to the net cross-section of the rod multiplied by the safe unit tension. $2\pi r'.t.q = \pi r'^2 f$, or $f r' = 2 q t$. As q is usually taken less than f; as with a square thread only half of the thickness is effective, and with a standard V thread quite a portion of the thickness must be deducted, nuts are usually given a thickness nearly or quite equal to the net diameter of the rod. Heads of bolts may be materially thinner.

135. **Eyebars.**—The eye for the connection of a tension bar to a pin is seen in Fig. 59. The pin is turned and the eye bored to a reasonably close fit. Since bearing first takes place at the back of the pin, the most intense pressure will be found there, and it will probably diminish at different points of the semi-circumference until the horizontal diameter is reached. The pressure on the pin may be found to extend slightly below that diameter. If these pressures are assumed to be normal, and are laid off in succession at 2′-2, 2-3, 3-4, . . . 5-6, and closed with 6-6′, the pull on the bar, a pole can be assumed at o and an equilibrium polygon or curve drawn, cutting the dotted centre line of the material of the eye about

as shown. By moving o horizontally and changing the point of beginning near A vertically, this equilibrium curve can be brought to the right position to satisfy the requirement that the sum of the products, from A to D, of each ordinate normal to the equilibrium curve multiplied by the force o-2, o-3, etc., there acting, shall equal zero. This requirement means that the tangents at A and at D, to the centre line, shall make the same angle with each other, before and while the pull is applied. See Greene's Graphics, Part II., Chap. VI.

It will be seen that the resultant force o-1 at the section at A is smaller than o-4 or o-6, the pull at B or C. Hence considerable deviation of the resultant force from the centre line at that section is not a serious matter. The eye was formerly made with an unnecessary enlargement at A, but is now commonly made circular through more than half of its perimeter. The edges of greatest stress are at A on the outside, B on the inside and C on the outside. This neck should be wide. The material in front of the pin within the dotted triangular area is of no service. In the looped eye of Fig. 59, made by bending the bar around the pin, that space is empty. Experiment has shown that for strength this loop should be long, from two to two and one-half diameters of the pin. If it were not for the weld and the excess of metal on either side of the pin, such a form of eye would be a satisfactory one.

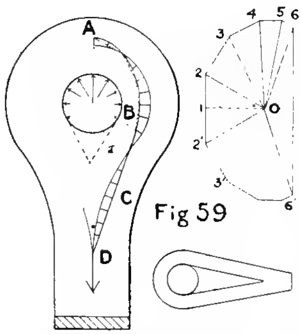


Fig 59

If y_0 is the deviation of the force from the centre of section at A or B, Fig. 59, the half width being y_1 and the pull o-1 or o-4 being P, the unit stress at the extreme edge, by § 127, will be

$$f = \frac{P}{S}\left(1 + \frac{y_0 y_1}{r^2}\right) = p_0\left(1 + \frac{6 y_0}{b}\right),$$

where b = width of eye on one side.

It is necessary to make the pin from $\frac{2}{3}$ to $\frac{3}{4}$ the width of the bar, in order to develop the strength of the latter, that is

to give sufficient bearing or compression area back of the pin. The right section through the eye exceeds that of the bar from 33 to 50 per cent.

Examples.—1. A round bolt $1\frac{1}{2}$ in. in diameter carries a load of 20,000 lbs. As its head is not square to its length, the centre of resistance is probably $\frac{1}{2}$ in. from the axis of the bolt. What is f in this case and how much greater than the mean stress? $f =$ 41,500 lbs. Since the elastic limit has been passed the actual maximum stress is probably less.

2. Find the maximum stress and the mean stress for a pull of 20,000 lbs. on a square eyebar $1\frac{1}{2}$ in. $\times$ $1\frac{1}{2}$ in. if the pin is $\frac{1}{2}$ in. out of centre. 26,670 lbs.

3. A rectangular bar, section 2 in. $\times$ 1 in., has a central pull of 8,000 lbs. Then $f =$ 4,000 lbs. If the bar is widened to 3 in. on one without change of force and its point of application, what is f?

4,667 lbs.

5-3 ..

CHAPTER IX.

COMPRESSION PIECES.—COLUMNS, POSTS AND STRUTS.

136. **Blocks in Compression.**—If the height of the piece is quite small as compared with either of its transverse dimensions, and the load upon it is centrally imposed, the load or force P may reasonably be considered as uniformly distributed over the cross-section S, and the unit stress f upon each square inch of section will be given by the formula

$$P = f\,S, \text{ or } f = P \div S,$$

as is the case with any tension member when the force is centrally applied.

137. **Load Not Central.**—So also, when the action line of the resultant load cannot be considered as central, but deviates from the axis of the piece a distance y_0, the force P can be replaced by the same force acting in the axis and a couple or moment Py_0, which moment must be resisted at every cross-section by a uniformly varying stress, forming a resisting moment exactly like that found at a section of a beam. Compare Fig. 58, and change tension to compression.

If f' is the uniform unit stress to resist the central load, so that $f' = P \div S$; and f'' is the unit stress at the extreme fibre which lies in the direction in which y_0 is measured, and at a distance y_1 from the axis,

$$Py_0 = f''\,I \div y_1, \text{ or } f'' = Py_0 y_1 \div I.$$

Hence the unit stress on the extreme fibre on the compression side, *i. e.*, on the side towards which y_0 is measured, will be, since $I = Sr^2$,

$$f = f' + f'' = \frac{P}{S} + \frac{Py_0 y_1}{I} = \frac{P}{S}\left(1 + \frac{y_0 y_1}{r^2}\right) = f'\left(1 + \frac{y_0 y_1}{r^2}\right).$$

The load that such a piece will carry is

$$P = \frac{fS}{1 + \frac{y_0 y_1}{r^2}}.$$

By comparison with the formula of the preceding section, it will be seen that the piece, when the load is eccentric, is weaker in the ratio $1 : 1 + \frac{y_0 y_1}{r^2}$.

The values of $y_1 S \div I$ or $y_1 \div r^2$ are given below for some of the common sections of columns, y_1 being measured in the direction h.

	I.	y_1.	S.	$y_1 S \div$ I.
Rectangle,	$\frac{bh^3}{12}$	$\frac{1}{2}h$	bh	$\frac{6}{h}$
Square,	$\frac{h^4}{12}$	$\frac{1}{2}h$	h^2	$\frac{6}{h}$
Circle,	$\frac{\pi d^4}{64}$	$\frac{1}{2}d$	$\frac{1}{4}\pi d^2$	$\frac{8}{d}$
Hollow Rectangle,	$\frac{bh^3 - b'h'^3}{12}$	$\frac{1}{2}h$	$bh - b'h'$	$\frac{6h(bh - b'h')}{bh^3 - b'h'^3}$
Hollow Circle,	$\frac{\pi(d^4 - d'^4)}{64}$	$\frac{1}{2}d$	$\frac{1}{4}\pi(d^2 - d'^2)$	$\frac{8d}{d^2 + d'^2}$

138. The Middle Third.—The mean or average unit stress is always found at the centre of the cross-section. When the maximum unit stress at the extreme fibre becomes twice the mean stress, the stress at the opposite edge, if the centre is in the middle of h, drops to zero. That is $f'' = f'$ or $y_0 y_1 \div r^2 = 1$. This will occur when $y_0 = r^2 \div y_1$, y_0 then being, for the rectangle, $\frac{1}{6}h$, and for the circle, $\frac{1}{8}d$. Hence, for a rectangular section in masonry, the centre of pressure must not deviate from the centre of figure more than one-sixth of the breadth in either direction, if the unit stress at the more remote edge is not to be allowed to become zero or tension. As masonry joints are supposed, in many cases, not to be subjected to tension in any part, the above statement is equivalent to saying that the centre of pressure or line of the resultant thrust must always lie within the *middle third* of any joint.

Likewise, for two cylindrical blocks in end contact, the centre of pressure should fall within the *middle fourth* of the diameter, if the pressure is assumed to be uniformly varying

and it is not permissible to have the joint tend either to open or to carry tension at the farther edge.

The unit pressure at the most pressed edge of a rectangle can be found for any deviation y_0.

1st. When the stress is over the whole joint, as before,

$$f = \frac{P}{S}\left(1 + \frac{y_0 y_1}{r^2}\right).$$

2d. When compression alone is possible, and only a part of the surface of the joint is under stress. The distance from the most pressed edge to the action line of P is $\frac{1}{2}h - y_0$. The entire pressed area $= 3(\frac{1}{2}h - y_0)b$, since the ordinates representing stresses make a wedge whose length along y is three times the distance of P from the most pressed edge.

$$P = \tfrac{1}{2}f \cdot 3(\tfrac{1}{2}h - y_0)b; \; f = \tfrac{2}{3}P \div (\tfrac{1}{2}h - y_0)b.$$

If the case is that of a wall, and P is the resultant force per unit of length, $b = 1$.

As y_0 increases, f increases, until finally the stone crushes at the edge of the joint, or shears on an oblique plane as described in § 23. Sometimes the pressure is not well distributed, from poor bedding of the stones, and *spalls* or chips, under the action of the shearing above referred to, may break off along the edge, without failure being imminent, since when the *high* spots break off others come into bearing.

P can never traverse one edge of the joint, if tension is not possible at the other edge, as the unit stress then becomes infinite. Some writers commit an error in determining the thickness of a wall by equating the moment of the overturning force about the front edge or toe with the moment of the weight of the wall about the same point. This process is equivalent to making the action line of P traverse that point. The centre of moments should be taken either at the outer edge of the middle third of the joint, when pressure is desired over the whole joint, or about a point at such a distance $\frac{1}{2}h - y_0$ from the front as will give maximum safe pressure at the front edge. A uniformly varying stress extending over three times that distance will equal P, as lately stated. A portion of the joint at the rear will then tend to open.

Examples.—A short, hollow, cylindrical column, 12 in. external diameter, 10 in. internal diameter, supports a beam which crosses the column 2 in. from its centre.

$$\frac{8d}{d^2 + d'^2} = \frac{8 \cdot 12}{144 + 100} = \frac{96}{244}. \quad p_1 = \frac{P}{S}\left(1 \pm \frac{2 \cdot 96}{244}\right)$$

$= p_0\,(1 \pm 0.8)$, or the stress at either edge will be 80% greater and less than the mean stress.

A joint, 10 ft. broad, of a retaining wall is cut at a point 3 ft. 9 in. from the front edge by the line of the resultant thrust above that joint. If this thrust per ft. of length of the wall is 28,000 lbs., the pressure per sq. ft. at the front edge will be $\frac{28,000}{10}(1 + 1\frac{1}{4} \cdot \frac{6}{10})$ $= 1\frac{3}{4} \cdot 2,800 = 4,900$ lbs. At the rear it will be 700 lbs. per sq. ft.

139. **Resistance of Columns.**—A column, strut or other piece, subjected to longitudinal pressure, is shortened by the compression. As perfect homogeneousness does not exist in any material, the longitudinal elements will yield in different amounts, so that there is apt to be a slight, an imperceptible tendency to curvature of the strut. Hence the action line of the applied load may not traverse the centres of all cross-sections of the piece. The product of the applied force into the perpendicular distance of its line of action from the centre of any cross-section will be a bending moment, which must develop a resisting moment at the cross-section, resulting in a varying stress, as in § 137. Equilibrium of the column as a whole will occur only when, for a given load, the axis of the column has assumed such a curve that, *at each cross-section, the resisting moment against lateral flexure equals the bending moment at that point due to the external force.* If the load is too great for that condition to be fulfilled, failure by flexure takes place.

140. **Remarks.**—This curvature under longitudinal pressure can be readily obtained with test specimens of most materials, even with some samples of cast-iron, and the form of the curve apparently conforms to the one to be deduced by theory. A *tendency* to such a curve must therefore exist under working stresses, although the curvature is imperceptible, unless the column happens to have its load perfectly axial, a contingency that cannot be safely relied upon. The

column formula, so-called, should therefore be confidently applied.

Further, as such curvature can be produced in test specimens not more than four or five diameters long, (Plate I, right figure, cast-iron; left figure steel), such a formula is applicable to columns and struts of any length. It is not necessarily to be applied, however, to very short posts, or blocks, for the relation $P = fS$ will determine their size with sufficient exactness.

141. The Yield Point Marks the Column Strength.—The influence of y_0 in determining the load a compression piece will carry has been shown in § 137 to be very marked. A column which has become sensibly bent under a load is very near complete failure. The moment of the load at the cross-section of greatest lateral deflection has then become so large that the stress on the extreme fibres passes the yield point, and the great increase of stretch at and above the yield point at once increases the bending moment greatly. Hence it is true that the yield point marks nearly the ultimate compressive strength of materials when tested in column form. § 149.

Again, the fact that, in tests of large columns, a *very slight shifting* of the point of application of the load at either end has a decided influence on the amount of weight such a column will carry is a confirmation of the statements with which this discussion opened. It also has a bearing upon the truth of the theoretical deduction as to the effect of eccentric loading, as discussed in § 137, and to be applied to long columns later.

142. Direction of Flexure.—The flexure usually occurs, unless there is some defect or weakness, in a direction *parallel* to the *least transverse dimension* of the strut, *i. e.*, perpendicular to that axis in the cross-section which offers the least resisting moment. By the application of longitudinal pressure to a slender rod its flexure may be made very apparent. The *form* of the column formula ought to resemble that of § 137,

$$P = fS \div 1 + \frac{y_0 y_1}{r^2}.$$

143. Formula for Columns. The Flexure Curve.—If the column is fixed in direction at its ends, by its connec-

tions to other pieces, or by having a broad, well-bedded base and cap, it will act in flexure much as a beam fixed at the ends. A couple or bending moment, which may be represented by M_0, will thus be introduced at each end. Let P = applied external force or load; v = any deflection ordinate, measured at right angles to the action line of P, from the original axis of the column to any point in the axis when bent; x = distance from one end along the original axis to any ordinate v; l = length of column.

The combination of the moment M_0 at the end of the column with the force P has the effect, as shown in Mechanics, of shifting the force P laterally a distance $M_0 \div P = v_0$; hence the action line of P is now parallel to the original axis, at a distance v_0 from it, or in the line F E of Fig. 60. The ordinate to the points of contraflexure is therefore v_0. This action can be more fully realized by conceiving that the bearing surface at A is removed, and that P acts at such a point on a horizontal lever as to keep the tangent to the curve at A strictly vertical.

As v is measured from the original axis, the bending moment at any section is $M = P(v_0 - v)$, which will change sign when the second term is larger than the first.

If the flexure is very slight, an equation similar to that used with beams may be written,

$$E\,I \frac{d^2v}{dx^2} = P\,(v_0 - v).$$

Multiply by dv, and integrate. As $\frac{dv}{dx} = 0$, when $v = 0$,

$$\frac{E\,I}{2}\left(\frac{dv}{dx}\right)^2 = P\,(v_0\,v - \tfrac{1}{2}\,v^2) + (C = 0).$$

When $\frac{dv}{dx} = 0$, at D, the middle of the length, v max. $= 2\,v_0$, or double the ordinate at the point of contraflexure F Let $P \div E\,I = B$.

The square root of the above equation gives

$$\frac{dv}{dx} = \sqrt{(B\,(2\,v_0\,v - v^2))}, \text{ or } dx = \sqrt{\left(\frac{1}{B}\right)}\frac{dv}{\sqrt{(2\,v_0\,v - v^2)}}.$$

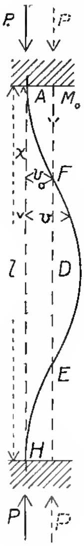
Integrate. As $v = 0$, when $x = 0$,

$$x = \sqrt{\left(\frac{1}{B}\right)} \left(\text{versin}^{-1} \frac{v}{v_0} + (C' = 0)\right).$$

$$v = v_0 \text{ versin } (x \sqrt{B}) = v_0 [1 - \cos (x \sqrt{B})].$$

As $1 - \cos \theta = 2 \sin^2 \frac{1}{2} \theta$, $v = 2 v_0 \sin^2 (\frac{1}{2} x \sqrt{B})$.

If, in this equation, $x = \frac{1}{2} l$, a value of v max. is obtained to be equated with the previous value $2 v_0$.



$$2 v_0 = 2 v_0 \sin^2 (\tfrac{1}{4} l \sqrt{B}), \text{ or } 1 = \sin^2 (\tfrac{1}{4} l \sqrt{B}).$$

$$\sin (\tfrac{1}{4} l \sqrt{B}) = 1, \text{ or } \tfrac{1}{4} l \sqrt{B} = \tfrac{1}{2} \pi, \text{ since } \sin^{-1} 1 = \tfrac{1}{2} \pi.$$

$$\text{As } B = P \div E I, \quad P = \frac{4 \pi^2 E I}{l^2},$$

which is commonly known as Euler's Formula. There fore the equation of the curve of the column is

$$v = 2 v_0 \sin^2 \left(\frac{\pi}{l} x\right).$$

To find the points of contraflexure, make $v = v_0$.

$$\sin \left(\frac{\pi}{l} x\right) = \sqrt{\tfrac{1}{2}} = \sin 45^\circ = \sin \frac{\pi}{4}.$$

$$\therefore \frac{x}{l} = \tfrac{1}{4} \text{ or } x = \tfrac{1}{4} l \text{ from either end.}$$

Hence the curve is made of four equal portions, A F, F D, D E and E H.

Fig. 60.

A column hinged, pin-ended, or free to turn at its ends, and of length represented by E F $= \frac{1}{2} l$, will have the same portion of stress at D that is due to *bending* as does a column of length A H $= l$, which is fixed at its ends.

If, in actual cases, F is considered to be practically in the same position horizontally as before loading, it may be said that a column fixed at one end and hinged at the other, of length F H $= \frac{3}{4} l$, will also have the same portion of stress arising from bending. The maximum deflection will then occur at one-third of its length from the hinged end. This result has been verified by direct experiment on a full-sized steel bridge member.

144. **Formula for Load.**—The load which a fixed column or strut will safely carry is determined by the maximum unit

stress on the extreme or outside particles on the *concave* side of the column at D, or at A and H, all three of which points are those of maximum unit stress. The unit stress here is limited to f, the maximum safe unit stress of the material, and the allowable load on a column is limited accordingly.

This unit stress f in the extreme fibres may be separated into two parts, one of which f' is the uniform compressive stress from P, or $f' = P \div S$, and the other is $f - f'$ the maximum compressive stress in the extreme fibres on the concave side from the bending moment Pv_0. Were there no direct compression, the post or strut could safely carry P as given by $P = 4\pi^2 E\, I \div l^2$; but as f' exists at the same time, the load P must be reduced in the ratio of the available compressive stress remaining to resist bending, or $f - f'$, to the max. safe stress f. Therefore multiply the value of P by this ratio $(f - f') \div f = 1 - \dfrac{f'}{f}$

$$P = \frac{4\pi^2 E\, I}{l^2}\left(1 - \frac{f'}{f}\right) = \frac{4\pi^2 E\, I}{l^2}\left(1 - \frac{P}{fS}\right).$$

$$P = \frac{fS}{1 + \dfrac{l^2 fS}{4\pi^2 E\, I}} = \frac{fS}{1 + a\dfrac{l^2}{r^2}},$$

where $r^2 = I \div S$, or what is known as the square of the radius of gyration, and $a = \dfrac{f}{4\pi^2 E}$, a quantity dependent upon the material. In practice, a is often greatly increased. See §§ 174–5. The last formula is known as Rankine's.

This value of a agrees very well with the one given by Rankine for iron, although the constants in the latter case were derived from experiments carried to failure. As such failure occurs at but little above the yield point, the difference in results will not be large. If $f = 36{,}000$, $4\pi^2 = 40$ approximately, $E = 28$ to $30{,}000{,}000$, $a = 1 \div 31{,}000$ to $33{,}000$.

Rankine gives $36{,}000\, S \div \left(1 + \dfrac{l^2}{36{,}000 r^2}\right)$.

145. Multipliers of a.—As seen above, $2l$ should theoretically be substituted for l when the column is hinged or free

to turn at its ends, in order to obtain the equivalent length of a column which is fixed at the ends; and for a column fixed at one end and hinged at the other $\frac{4}{3}l$ should be substituted for l, for the same reason. Or, more conveniently, a may be used for a column fixed at the ends and of length l; $4a$ for a column hinged at both ends and of length l; and $\frac{16}{9}a$ for a column hinged at one end and fixed at the other, length l. Actual tests, carried however to the extreme of bending or crippling, appear to show that a column bearing on a pin at each end is not hinged or perfectly free to turn; hence the multipliers of a more commonly used, instead of 1, $\frac{16}{9}$ and 4, are 1, $\frac{4}{3}$ and 2.

The theoretical ratio of strength of a column hinged at ends to that of one fixed at ends is

$$\left(1 + a\frac{l^2}{r^2}\right) \div \left(1 + 4a\frac{l^2}{r^2}\right) = \frac{r^2 + al^2}{r^2 + 4al^2}.$$

If $l = 100r$, the ratio becomes $\dfrac{1 + 10,000a}{1 + 40,000a}$; and, if $a = \dfrac{1}{36,000}$, it becomes $\dfrac{46,000}{76,000}$, or 23 to 38, about 0.6. A column fixed at ends is, for the above values, some sixty-five per cent., or two-thirds stronger than one hinged at ends.

146. Pin Friction.—Some regard columns as neither perfectly fixed nor perfectly hinged, and use but one value of a for all, which might perhaps then be taken as a mean value. The moment of friction on a pin is considerable. If P is the load on a post or strut, d the diameter of the pin, and $\tan\varphi$ the coefficient of friction of the post on the pin, the moment of friction at the pin will be $P \cdot \frac{1}{2}d \cdot \tan\varphi$; and this moment, if greater than $M_0 = Pv_0$, will keep the post restrained at the end, so that the tangent there to the curve remains in its original direction. As P and v_0 increase, M_0 will become the greater when v_0 exceeds $\frac{1}{2}d \tan\varphi$; the column will then be imperfectly restrained at its ends, and the inclination will change. As the friction of motion is less than that of rest, such movement when started, may be rapid. Some tested columns, showing at first the curve of Fig. 60, known as triple flexure, have suddenly sprung into a single curve and at once offered less resistance.

It may be doubted whether ordinary columns, under working loads, ever develop a value of v_0 sufficient to overcome the pin friction, unless the column is very slender and the diameter of the pin small.

147. **Failure by Tension.**—In rare cases, when the material has but moderate tensile strength, as compared with the compressive strength, for example cast-iron, and the column is very slender, the convex side may be the weaker. The maximum allowable unit tension arising from the bending moment will be $-(f + f')$, reducing to an actual unit tension $-f$ when the uniform compression f' is combined with it. The formula for P then becomes

$$P = \frac{4\pi^2 E I}{l^2}\left(\frac{f'}{f} + 1\right) = \frac{fS}{a\frac{l^2}{r^2} - 1}.$$

148. **Short Columns and Slender Columns.**—If a column is very short, the second term in the denominator of the value of P, § 144, becomes very small, and the formula practically reduces to $P = fS$. Some engineers use this form for iron and steel built struts up to a limit of $l = 60r$ or $80r$.

On the contrary, if the column is extremely slender, the second term of the denominator becomes so much larger than unity as to practically overpower the first term, and the expression becomes Euler's Formula, § 143. In that case the stress on the extreme fibre from the moment of flexure far outweighs that from the direct thrust, and the latter may be neglected.

Example. —A wrought iron column of hollow cylindrical form, 20 ft. long, not fixed at ends, must carry a static load of 36,000 lbs. If $f = 10{,}000$ lbs.; $4a = \frac{4 \cdot 10{,}000}{4\pi^2 \cdot 28{,}000{,}000} = \frac{1}{28{,}000}$; and the mean diameter of a thin ring is d; then by VI., § 99, $r^2 = \frac{1}{8}d^2$, and $36{,}000 \dfrac{10{,}000\,\pi dt}{1 + \dfrac{240 \quad 240 \cdot 8}{28{,}000\, d^2}}$, or $3.6\left(1 + \frac{16.5}{d^2}\right) = \frac{22}{7}dt.$

If $t = \frac{1}{4}$ in., $\frac{11}{14}d^3 - \frac{36}{10}d^2 = 59.4$. Solve for d by trial. As d is between 6 and $6\frac{1}{2}$ in., a cylinder of 6 in. interior diam. and $\frac{1}{4}$ in. thick will be satisfactory.

149. **Experimental Results.**—The crippling strengths of different columns have been obtained experimentally for various lengths and values of S and r. When such results are plotted with ordinates $P \div S = f$ and abscissas $l \div r$, and

curves are drawn to agree as nearly as may be with the mean locus of the more or less scattered points thus found, it is not surprising that the equations of such curves do not agree very closely with the one deduced in § 144. The general trend of the area covered by the points is, however, reasonably satisfactory. The same reason for variation applies here as in the deduction of f from tests of beams loaded to rupture, in which tests a value of f is obtained differing from the ultimate tensile and compressive strength of the material. The assumption of plane sections and uniformly varying stress will be true only below the elastic limit, and only within that limit should f be used. The formulas are intended for the determination of sections for working, not breaking, loads.

Still, as already pointed out, there is not a wide disagreement between experimental results and those given by the formula. For the actual strength of iron and steel in compression, when used in long struts, is little, if any, above the yield point (as may be seen from the lowest curve of the diagram, Fig. 1). Such struts, if slightly bent under a load, fail rapidly when the load is increased by a small amount As a slight error in centering the experimental column has a marked influence on its fibre stress, by the introduction of a moment Py_0, some of the scattered results of tests may be attributed to such a cause.

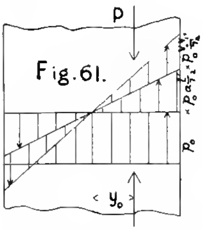


Fig. 61.

150. Swelled Columns.—Some posts and struts, especially such as are built up of angles connected with lacing, are swelled or made of greater depth in the middle. If the strut is perfectly free to turn at the ends, such increase in the value of r^2 may be quite effective, and r^2 for the middle section may be used in determining the value of P, provided the latter does not too closely approach the uniform compression value at the narrower ends. But if the strut is fixed at the ends, or is attached by a pin whose diameter is large enough to make a considerable friction-moment, such enlargement at the middle is useless; for an equally large value of r^2 ought to be found at the ends also. Hence swelled columns and struts are but little used.

151. Column Eccentrically Loaded.—When the load is applied eccentrically to a long column, the maximum unit

stress found in the extreme fibre on the concave side must be due to three combined effects:—

1st. The stress due to the load P, or $p_0 = P \div S$. Fig. 61.
2d. The stress due to the resisting moment set up by Py_0.
3d. The stress due to the resisting moment set up by Pv_0.

$$\text{From § 137, } f = \frac{P}{S}\left(1 + \frac{y_0 y_1}{r^2}\right) = p_0 + p_0 \frac{y_0 y_1}{r^2}.$$

$$\text{From § 144, } f = \frac{P}{S}\left(1 + a\frac{l^2}{r^2}\right) = p_0 + p_0 a \frac{l^2}{r^2}.$$

If then the column is long and the line of action of P deviates from the original axis of the column by a distance y_0, the three expressions, p_0, $p_0 y_0 y_1 \div r^2$ and $p_0 a l^2 \div r^2$ should be added, giving

$$f = p_0\left(1 + a\frac{l^2}{r^2} + \frac{y_0 y_1}{r^2}\right). \quad \therefore \quad P = \frac{fS}{1 + a\dfrac{l^2}{r^2} + \dfrac{y_0 y_1}{r^2}}.$$

Since y_0 will determine the direction of flexure, r must be taken in this case in the direction y_0; that is, the moment of inertia and r must be obtained about the axis through the centre of gravity and lying in the plane of the section, perpendicular to y_0.

That the moment Py_0, although small, has a decided weakening effect on a column is proved by experiment, and its unintended presence may explain some anomalies in tests. See "Experiments on Strength of Wrought Iron Struts," by James Christie, Trans. Am. Soc. C. E., Vol. XIII., April, 1884.

Example.—If, in the example of § 148, the load is applied $\frac{1}{2}$ in. off centre, $f = \dfrac{36{,}000 \cdot 7 \quad 4}{6 \cdot 22}\left(1 + \dfrac{240 \cdot 240 \cdot 8}{28{,}000 \cdot 36} + \dfrac{3 \cdot 8}{2 \cdot 36}\right)$ $= 7{,}650\ (1.8) = 13{,}800$ lbs. The stress on the convex side is $+ 1{,}530$.

152. Straight-Line Formula for Columns.—Rankine's formula $P \div S = f \div \left(1 + a\dfrac{l^2}{r^2}\right)$, when plotted for various values of $P \div S = y$, and $l \div r = x$, gives a rather flat reversed curve, as shown in Fig. 62. The first part of the curve is not of practical value, as columns which have small

values of $l \div r$ can be determined with sufficient accuracy by $P = f S$. The largest values of $l \div r$ also are of no service, as the curve yields too small values of f for use, and approaches the quantity $P \div S = 4\pi^2 E r^2 \div l^2$. The equation of a straight line which shall nearly coincide with the flattest portion of the curve, on either side of the point of contraflexure, may be substituted for Rankine's formula for convenience, and the same values of mean stress will be practically obtained within working limits of $l \div r$.

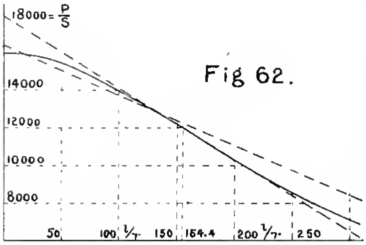


Fig 62.

Rankine's formula, upon the substitution of the symbols x and y as above, may be written $y = f \div (1 + ax^2)$, and the co-ordinates x' and y' of the points of contraflexure found by putting $\frac{d^2y}{dx^2} = 0$. Thus

$$\frac{dy}{dx} = \frac{-2afx}{(1 + ax^2)^2}; \quad \frac{d^2y}{dx^2} = \frac{-2af(1 + ax^2) + 8a^2fx^2}{(1 + ax^2)^3} = 0.$$

Dropping factors, $-(1 + ax^2) + 4ax^2 = 0 \therefore 3ax^2 - 1$; $x' = 1 \div \sqrt{3a}$, and $y' = \frac{3}{4}f$, the desired co-ordinates.

The equation of a straight line tangent to the curve at the point of contraflexure may now be deduced from the well known formula $y - y' = \frac{dy'}{dx'}(x - x')$.

$$\frac{dy'}{dx'} = \frac{-2af}{\sqrt{3a}(\frac{4}{3})^2} = -\frac{3}{8}f\sqrt{3a}.$$

$$y = \frac{3}{4}f - \frac{3}{8}f\sqrt{3a}(x - \frac{1}{\sqrt{3a}}) = \frac{9}{8}f - \frac{3}{8}f\sqrt{3a} \, . \, x.$$

If $f = 16{,}000$, $E = 29{,}000{,}000$ and $a = f \div 4\pi^2 E$,

$P \div S = f' = 18{,}000 - 39\frac{l}{r}$ for fixed ends, and

$$f' = 18{,}000 - 78\frac{l}{r} \text{ for hinged ends.} \qquad (1.)$$

The curve of Fig. 62 is plotted for these values of f and E, and the tangent at the point of contraflexure is shown for a column with fixed ends.

If $f = 14,000$, and E and a are as before, there result two corresponding values of the mean stress, for fixed and for hinged ends.

$$f' = 15,750 - 32\frac{l}{r} \text{ and } 15,750 - 64\frac{l}{r}. \qquad (2.)$$

If it is desired to use a straight line formula for moderately short columns, such a line might be drawn through $x'' = l \div r = 0$ and $y'' = P \div S = f$ and through the point of contraflexure. Since

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'),\ y - \tfrac{3}{4}f = \frac{f - \frac{3}{4}f}{0 - \frac{1}{\sqrt{3a}}}\left(x - \frac{1}{\sqrt{3a}}\right),$$

$$y = f - \tfrac{1}{4}f\sqrt{3a}\,.\,x.$$

If $f = 16,000$, and E and a are as above,

$$f' = 16,000 - 26\frac{l}{r} \text{ for fixed ends, and}$$

$$= 16,000 - 52\frac{l}{r} \text{ for hinged ends.} \qquad (3.)$$

A straight line through $l \div r = 0$, $P \div S = 16,500$, and the point of contraflexure is shown in the figure, and yields

$$f' = 16,500 - 27\frac{l}{r}, \text{ and } 16,500 - 54\frac{l}{r}. \qquad (4.)$$

If the safe unit stress for live load is taken as one-half its value for dead load, the preceding equations (1.) to (4.) would become for live load:—

Fixed ends.	Hinged ends.	
$9,000 - 14\frac{l}{r}$	$9,000 - 27\frac{l}{r}$	(1'.)
$7,875 - 11\frac{l}{r}$	$7,875 - 22\frac{l}{r}$	(2'.)
$8,000 - 9\frac{l}{r}$	$8,000 - 18\frac{l}{r}$	(3'.)
$8,250 - 10\frac{l}{r}$	$8,250 - 20\frac{l}{r}$	(4'.)

It will be seen that the first term in each case is obtained by dividing by two; but the second term must be computed from $f\sqrt{3a}$.

If the value of f for combined dead and live load, when they are nearly equal, is taken as 12,000 lbs., there will result in place of (3.) and (3'.),

$$12{,}000 - 17\frac{l}{r} \text{ and } 12{,}000 - 34\frac{l}{r} \qquad (5.)$$

Such modifications of Rankine's formula are convenient for use and are often specified.

For current formulas see § 175.

If $f = 1{,}000$ and $E = 1{,}400{,}000$, there will be obtained, in place of (3.), $1{,}000 - 2l \div r$. But, as timber struts are usually rectangular, $12r^2 = h^2$, where h is the least dimension, and there results

$$f' = 1{,}000 - \frac{6l}{10h} \text{ for fixed ends.}$$

The subtractive term is much smaller than the one commonly employed. See § 170.

153. Transverse Force on a Column.—The resisting power of a column or strut to which a transverse force is applied in addition to the load in the direction of the axis, and the proper dimensions of the strut, are involved in some doubt. Theoretically, the formula is deduced as follows:—

From the formula for the resisting moment of a beam, $M = f'' I \div y_1$, the stress on the outer layer from such bending moment is $My_1 \div I$. Hence, if M is *that particular value* of the bending moment (from the transverse load or force) which exists at the section of maximum strut deflection, *where the column stress is greatest* (that is, at the middle for a column with hinged ends, but perhaps at the ends for a column with fixed ends, since M may then be greater at the ends), the max. unit stress on the concave side

$$f = \frac{P}{S}\left(1 + a\frac{l^2}{r^2}\right) + \frac{My_1}{I}; \text{ and}$$

$$P = \frac{\left(f - \dfrac{My_1}{I}\right)S}{1 + a\dfrac{l^2}{r^2}} = \frac{fS - \dfrac{My_1}{r^2}}{1 + a\dfrac{l^2}{r^2}}.$$

when the max. fibre stress does not exceed f. Or, it may be said that, at the section in question, P is moved laterally by the moment M a distance $y_0 = M \div P$. Then by § 151,

$$P = \frac{fS}{1 + a\frac{l^2}{r^2} + \frac{My_1}{Pr^2}}, \text{ or } P\left(1 + a\frac{l^2}{r^2}\right) + \frac{My_1}{r^2} = fS;$$

$$\text{whence } P = \frac{fS - \frac{My_1}{r^2}}{1 + a\frac{l^2}{r^2}}.$$

The value of r^2 to be used in this formula is that for flexure in the plane of M. It may be convenient, when a strut of two channels or similar built up members is proposed, to determine the depth and section of the channels to give the necessary *column strength* alone, for r^2 about an axis perpendicular to their webs, and then to place them a sufficient distance apart in the plane of the moment M to give a value of r^2 which will also satisfy the above formula for P.

Strut-beams are not economical, and their introduction should be avoided, if possible.

Example.—A steel column of two 9 in., $20\frac{1}{2}$ lb. channels, laced together; combined section 12 sq. in., I for axis perpendicular to web $= 2 \times 58.5$, $r = 2.72$, $l \div r = 90$. If the column is 20 ft long between pins, its supporting capacity, by (4.) § 152, will be $P = 12(16{,}500 - 54 \quad 90) = 139{,}700$ lbs.

If a horizontal force of 14,000 lbs. is applied at 4 ft. from one end, the distance which these channels should be spaced apart, in the plane of the bending moment, if the flanges are turned in, is about 20 in. For I of one channel about its own axis parallel to web = 2.5; distance of axis from back = 0.56 in.; I for column with 20 in. spacing $= 2[2.5 + 6(10 - 0.56)^2] = 1{,}191$; $r^2 = 99.2$; $l \div r = 24$. $P \div S = 11{,}640 = f' - 1{,}300$; $f' = 12{,}940$ lbs. M at middle of column $= (14{,}000 \quad 4 \div 20)120 = 336{,}000$ in. lbs. $f'' = 336{,}000 \,.\, 10 \div 1{,}191 = 2{,}820$ lbs. $f' + f'' = 15{,}760$ lbs.

154. Lacing Bars.—The parts of built-up posts are usually connected with lacing straps or bars. See XVI., Plate III. These bars carry the shear due to the bending moment arising from the tendency of the post to bend, and should be able to stand the tension and compression induced by the shear. At the

ends and where other members are connected, in order to insure a distribution of load over both members, batten or connection plates at least as long as the transverse distance between rivet rows are used in the best practice.

The value of this shear would be found by taking the first derivative of the bending moment, Pv; but v is unknown. As the lacing bars are usually of uniform size throughout the length of the post, it will suffice to determine their dimensions at the ends of the strut.

From the formula $f = \frac{P}{S}\left(1 + a\frac{l^2}{r^2}\right)$ it appears that the max. unit stress due to tendency to bend is $\frac{P}{S}\,a\,\frac{l^2}{r^2}$, which may be equated with $My_1 \div I$. It may be assumed that the moments at the several points along the axis of a strut with fixed ends vary as do those of a beam loaded with W at the middle and fixed at the ends. In the latter case, by § 117, $M = \frac{1}{8}Wl$ at the two ends and at the middle, and the points of contraflexure are at $\frac{1}{4}l$ from either end. Hence the two curves must be much alike. Then

$$\frac{P}{S}\,a\frac{l^2}{r^2} = \frac{Wly_1}{8\,I} \text{ or } W = \frac{8\,Pal}{y_1}.$$

As the maximum shear in the above case is $\frac{1}{2}W$,

$$F \text{ max.} = \tfrac{1}{2}W = \frac{4\,Pal}{y_1}.$$

If the strut has hinged ends, replace a by $4a$ or $2a$, as explained in § 145.

Example.—The 9 inch column of the last section, if loaded with 140,000 lbs. and spaced 9 in., so that the lacing bars are 8 in. long between rivets, will have $a = 16{,}500 \div (40 \cdot 29{,}000{,}000) = 1 \div 70{,}000$ nearly. Then $F = 4 \cdot 4 \cdot 140{,}000 \cdot 240 \div (4\frac{1}{2} \cdot 70{,}000) = 1{,}700$ lbs. Pull or thrust in bar is about 2,000 lbs. The shearing or bearing value of one $\frac{5}{8}$ rivet in $\frac{1}{4}$ in. plate is 2,300 lbs. A 2 in. × $\frac{1}{4}$ in. bar in tension, net section $\frac{5}{16}$ sq. in. carrying 2,000 lbs. will have a unit stress of at least 6,400 lbs. In compression it can carry some 5,000 lbs.

If this column resists the horizontal force of 14,000 lbs. also, the shear will be $770 + 2{,}800 = 3{,}570$ lbs. at one end and 12,000 lbs. at the other. The bars will be some 21 in. long, and must be crossed and riveted at intersections. A continuous plate may be advisable where the shear is heavy.

Each piece should be of equal strength throughout all its details. A post or strut composed of two channels, connected by lacing bars and tie plates, is proportioned for a

certain load, the mean unit stress being reduced in accordance with the formula in which the variable is the ratio of the length to the least radius of gyration of the *whole section*. In the lengths between the lacing bars, this ratio for *one channel* with its own least radius should not be greater than for the entire post. Nor should the *flange* of the channel have any greater tendency to buckle than should one channel by itself. The same thing applies to the ends of posts, where flanges are sometimes cut away to admit other members. Quite a large bending moment may be thrown on such ends, when the plane of a lateral system of bracing does not pass through the pins or points of connection of the main truss system.

A recent specification reads: Single lacing bars shall have a thickness of not less than $\frac{1}{40}$, and double bars, connected by a rivet at their intersection, of not less than $\frac{1}{60}$ of the distance between the rivets connecting them to the member. Their width shall be— For 15 in. channels, or built sections with $3\frac{1}{2}$ and 4 in. angles, $2\frac{1}{2}$ inches, with $\frac{7}{8}$ in. rivets; for 12 or 10 in. channels, or built sections with 3 in. angles, $2\frac{1}{4}$ inches, with $\frac{3}{4}$ in. rivets; for 9 or 8 in. channels, or built sections with $2\frac{1}{2}$ in. angles, 2 inches, with $\frac{5}{8}$ in. rivets.

The distance between connections of the lacing bars shall not exceed eight times the least width of the segments connected.

All segments of compression members, connected by lacing only, shall have ties or batten plates placed as near the ends as practicable. These plates shall have a length of not less than the greatest depth or width of the member and shall have a thickness of not less than $\frac{1}{50}$ of the distance between the rivets connecting them to the compression member.

Examples.—1. A single angle-iron, $6 \times 4 \times \frac{3}{8}$ in. and 6 ft. 8 in. long, is in compression. Use $r = 0.9$ or obtain it from § 99, X. $S = 3.61$ sq. in. If $P \div S = 12{,}000 - 34l \div r$, what will it carry? 32,400 lbs.

2. A square post 16 ft. long is expected to support 80,000 lbs. If $f = 1{,}000$ and the subtractive term is $2l \div r$, what is the size? 10×10 in.

3. A cylindrical rod of steel is 4 in. in diam. and 100 in. between pins. What is the max. allowable thrust if $f = 8{,}000$ lbs. for an alternating load and $(3')$ is used? 78,000 lbs.

4. What is the safe load on a hollow cylindrical cast-iron column 13 ft. 6 in. long, 6 in. external diam. and 1 in. thick, if it has a broad, flat base, but is not restrained at its upper end? $f = 9{,}000$ lbs., $E = 17{,}000{,}000$, $S = 15.7$ sq. in.

124,000 lbs.

5. If a short wooden post, 12 in. square, carries 28,800 lbs. load, and the centre of pressure is 3 in. perpendicularly from the middle of one edge, what will be the max. and the mean unit pressure, and the max. unit tension, if any?

500 lbs.; 200 lbs.; — 100 lbs.

CHAPTER X.

SAFE WORKING STRESSES.

155. **Endurance of Metals Under Stress.**—It is importtant to determine how long a piece may be expected to endure stress, when constant, when repeated, or when varied and perhaps reversed; and it is still more important to find what working stresses may be allowed upon a given material in order that rupture by the stresses may be postponed indefinitely.

The experiments carried on by Wöhler* and Spangenberg, and afterwards continued by Bauschinger, show the action of iron and steel under repeated stress.

156. **Woehler's Experiments.**—A number of bars of wrought iron and steel were subjected to a load which repeatedly varied between zero and a certain quantity. One series of tests was by direct tension, another series by transverse bending. When the bar broke under the load, a similar bar was tested under a reduced load, and was found to resist a greater number of applications before fracture. Under a smaller load the number of applications necessary to produce fracture was found to increase. Finally a load was reached which did not cause failure after some 40 million repetitions. This last value was taken as the limiting safe strength for loads applied in that particular way. Thus bars of unhardened Krupp spring steel under a steady load broke with 110,000 lbs. tension per sq. in.; with 100,000 lbs., fracture ensued after 40,000 repetitions; with 90,000 lbs. after 72,000 repetitions; 80,000, 70,000 and 60,000 lbs. required 132,000, 197,000 and 468,000 repetitions; and for 50,000 lbs. the bar

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* Wöhler, Uber die Festigkeitsversuche mit Eisen and Stahl, Berlin, 1870. Zeitschrift für Bauwesen, Vols. X, XIII., XVI., XX.
See also Burr's Elasticity and Resistance of Materials.
Technic, Univ. of Mich., 1889.

endured 40 million repetitions of load from zero without fracture.

In other experiments the stresses were varied between different limits and from tension to compression.

For specimens taken from a certain iron axle, experiments showed that alternations of stress between the following limits of tensile (—) and compressive (+) stress per square inch, might take place with equal security against rupture with several million repetitions.

— 15,500 lbs.	and	+ 15,500 lbs.	Difference	31,000 lbs.
— 29,000	"	0	"	29,000 "
— 43,000	"	— 23,500 lbs.	"	19,500 "

it being of course assumed that in all cases the maximum stress is less than that required to produce rupture under a static load.

Also unhardened spring steel gave the limits

— 48,500 lbs.	and	0	Difference	48,500 lbs.
— 68,000	"	— 24,000 lbs.	"	44,000 "
— 78,000	"	— 39,000 "	"	39,000 "
— 87,500	"	— 58,500 "	"	29,000 "

For specimens from a Krupp cast-steel axle the following values gave equal security against rupture.

— 27,000 lbs.	and	+ 27,000 lbs.	Difference	54,000 lbs.
— 47,000	"	0 "	"	47,000 "
— 78,000	"	— 34,000 "	"	44,000 "

and for shearing resistance of a cast steel axle, first in one direction and then in the opposite direction,

— 21,500 lbs.	and	+ 21,500 lbs.	Difference	43,000 lbs.
— 37,000	"	0	"	37,000 "

157. **Woehler's Laws.**—Wöhler's laws, deduced from many and long-continued experiments, are

Rupture of material may be caused by repeated alternations of stress, none of which attains the absolute breaking limit.

Differences of stress, the *extreme range* of stress, are a sufficient cause of rupture; and the absolute magnitude of ex-

treme stress is controlled by the fact that, as the stress increases, the differences which are sufficient to cause rupture become less.

The above examples indicate this law.

158. Bauschinger's Experiments.—In endurance tests made by Professor Bauschinger specimens were repeatedly pulled from zero to certain stresses. Many of the specimens were several times examined for elastic limit during the progress of the endurance tests. The stresses to which bars were subjected were varied also in *kind* and amount, and the number of applications required to produce fracture, when fracture occurred, were recorded. It was found that, in all cases, the number of repetitions of loading borne by the bar diminished with increased range of variation of stress, and the diminution was surprisingly regular.

He derived the following values of the highest tensile stress which may be applied to each material *indefinitely*, beginning each pull at zero, or what may be termed the *maximum safe tensile stress*, for working stress from zero. There is also given the maximum permissible range of stress, when the extremes are of opposite kinds, and when they are of the same kind. See p. 160.

To these values are added the elastic limit, yield point, and breaking stress for a single static application, all in pounds per square inch. Values for different samples will not agree exactly, and must be considered as approximate averages.

If a piece has applied to it a tensile stress which reaches the yield point, it can, after a period of rest, have applied to it with safety an equal stress in compression. When, however, the stresses repeatedly alternate between plus and minus, the algebraic difference between them, the numerical sum must be taken into account, and the evidence seems to be that such sum cannot, for continued repetitions of stress, ever safely exceed the distance between zero and the yield point.

To repeat a previous statement:—*Elastic limit* marked that load at which either permanent set could first be detected, or where increments of stress and extension ceased to be proportional to each other, the earliest certain and continued indication either way being taken to mark that limit.

The first indication of permanent set was very strictly noted as the elastic limit. *Yield point* marked that load at which rapid and considerable yield first took place under a steady load, and was the point called elastic limit in ordinary testing, done without very accurate means for measuring small extensions. The original elastic limit was far below yield point in many of the specimens, notably so in the iron plate, where the difference is fifty per cent. of the yield point, and in the steel plate where the difference is twenty per cent.

BAUSCHINGER'S ENDURANCE TESTS.

Stresses requiring five to ten million repetitions to cause fracture, in pounds per square inch.

MATERIAL.	OPPOSITE STRESSES.		ONE STRESS ZERO.		SIMILAR STRESSES.	
	Least.	Greatest.		Greatest.	Least.	Greatest.
Wrought Iron	— 16,000	+ 16,000	0	29,000	25,500	43,000
Bar Iron	— 17,500	+ 17,500	0	32,000	30,000	49,000
	— 19,000	+ 19,000	0	35,000	29,000	49,000
Mild Steel Plate	— 19,000	+ 19,000	0	35,000	32,000	53,000
Steel Axle	— 23,000	+ 23,000	0	43,000	44,000	71,000
Steel Rail	— 21,000	+ 21,000	0	43,000	43,000	69,000
Mild Steel Boiler Plate	— 19,000	+ 19,000	0	35,000	29,500	50,500

MATERIAL.	ELASTIC LIMIT.	YIELD POINT.	ULTIMATE STATIC.	ELONGATION PER CENT.
Wrought Iron Plate	15,000	30,000	51,000	15.5
Bar Iron	26,000	32,500	57,500	12.5
	32,000	35,000	57,000	14.
Bessemer Mild Steel Plate	34,000	42,000	62,000	21.
Thomas Steel Axle	38,000	47,000	87,000	18 5
Thomas Steel Rail	40,000	43,500	85,000	19 5
Thomas Steel Boiler Plate	37,000	42,000	58,500	26.

The steel was somewhat irregular.

159. Maximum Tensile Stress.—The maximum safe tensile stresses from zero, regarded as more or less rough approximations, show that every such stress is near and below the yield point. Hence the *limiting safe stress on sound bars* of iron and steel, for tensions alternating from zero to that stress, is just a little less than the original yield point of the metal.

160. Elastic Limit and Alternating Stresses.—It appears that a bar subjected to alternate pressure and tension will break after a sufficient number of repetitions with a stress less than its primitive elastic limit. But it was found that the application of a stress exceeding the elastic limit raised that limit,—in certain cases, nearly to the breaking stress. Prof. Bauschinger, in 1886, said that tension greater than the true elastic limit applied to a bar raised its elastic limit in tension, but not without lowering the limit in compression, and *vice versa*. Even a moderate raising of the tension limit may lower the compression limit to zero. Therefore the law that the elastic limit can be raised by stress does not apply to alternating stresses. Further, these artificially produced elastic limits are extremely unstable. He advanced the view that the primitive elastic limit of many materials is an artificially raised one. The material has been subjected to mechanical operations in manufacture which are equivalent to straining actions. Alternate compression and extension has the effect of raising an artificially lowered, or lowering an artificially raised, elastic limit. By subjecting a bar to a few alternations of equal stresses, which are equal to or somewhat exceed the elastic limits, the latter tend towards fixed positions, which Bauschinger called the *natural* elastic limits. The range of stress for which a bar is perfectly elastic after a few repetitions of such alternating stresses appears to agree very closely with Wöhler's range of stress for unlimited repetitions of alternating stresses.

A curious discovery, that moderate hammering on the end of cast-iron bars appears to relieve internal stresses, perhaps cooling stresses, and to increase the tested strength, seems to point in the same direction.

161. Alteration of Structure.—In discussing the question of internal changes, Unwin says:* It would appear likely that any gradually progressive alteration or fatigue of the bar would be manifested in some way in alteration of the strength, the elastic limit, or the elongation of the bar when tested in the ordinary way. But this, so far, appears not to

*The Testing of Materials of Construction: Unwin.

be the case. A bar subjected to so many repetitions of loading that it is known to be on the point of breaking, or a piece of a bar already broken in an endurance test, gives in the testing machine no indications that the strength or ductility has been altered. See §166, (*l*.). It is in accordance with experiments on pieces of structures long subjected to loading; and no one would guess that these test pieces were in any respect different from new material. But the fact still remains that material subjected to repeated loading is different from new material. The material, after a certain number of repetitions with a given range of stress, does break with fewer subsequent repetitions.

Whatever the alteration produced by repetition may be, it certainly does not appear to be a loss of strength (statical resistance). If it is a loss of ductility or power of elongation, it must be a loss confined to very short portions, or planes of weakness, in the bar; for, if not, it would be shown in ordinary testing. In certain cases flaws or fissures have been found to be present in bars subjected to so many repetitions of load that they were on the point of breaking. It is at least conceivable that repetition of stress picks out sections of weakness in the bar, and that the deterioration is confined almost to such planes. The deterioration may be primarily a loss of power of yielding in the particles near the plane of weakness, and not a loss of tenacity. Such a loss of ductility at a section might well show itself finally in a rapidly spreading fissure or crack. This explanation is purely hypothetical, but it is at least in accordance with a curious fact. Bars fractured in the Wöhler machines usually showed no trace of drawing out, however ductile the material might be, tested statically. They broke as if the material had been perfectly brittle. This peculiar fracture, without indication of any plastic drawing-out, is not uncommon in fractures of tires, axles and other structures in ordinary practice.

162. Launhardt-Weyrauch Formula.—Two formulas have been advanced, based on Wöhler's experiments, for the determination of the allowable unit stress of either sign on any material when the range as well as the magnitude of the stress is considered. Launhardt proposed the following

formula for the breaking load of a member which is subjected to (apparent) stresses, varying between maximum and minimum stress of the same kind:

$$a = u\left(1 + \frac{t - u}{u} \cdot \frac{\text{min. stress}}{\text{max. stress}}\right),$$

in which a is the breaking weight or stress under the conditions to which the member is subjected, u is the greatest stress which the piece will bear without breaking, if repeated from zero to u an indefinite number, many million, of times, and t is the breaking stress at a single application.

Weyrauch extended Launhardt's formula, to cover cases of alternate tension and compression, in which case *min. stress* is to be considered as negative, giving

$$a = u\left(1 + \frac{u - v}{u} \cdot \frac{\text{min. stress}}{\text{max. stress}}\right),$$

where v is the greatest stress which the piece will bear without breaking, if repeated from $+ v$ to $- v$ an indefinite number of times.

In some of Wöhler's experiments u appears to be greater than $\frac{1}{2}\,t$, and to approximate $\frac{2}{3}\,t$. This value is assumed by Weyrauch and gives $(t - u) \div u = \frac{1}{2}$. Also v by Wöhler's experiments is equal to $\frac{1}{3}\,t$, or $(u - v) \div u = \frac{1}{2}$. Therefore both formulas become

$$a = u\left(1 + \frac{\text{min. stress}}{2\ \text{max. stress}}\right),$$

the sign of the second term changing when the minimum stress and maximum stress have opposite signs.

When a is the maximum stress, as it must be in designing a member which is to be subjected to minimum stress also,

$$\text{Max. stress} = u + (t - u) \cdot \frac{\text{min. stress}}{\text{max. stress}}, \text{ or}$$

$$\text{Max. stress}^2 - u \text{ max. stress} + \tfrac{1}{4}\,u^2 = \tfrac{1}{4}\,u^2 + (t - u) \text{ min. stress}.$$

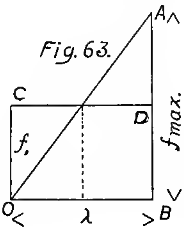
$$\text{Max. stress} = \tfrac{1}{2}\,u + \sqrt{\left(\tfrac{1}{4}\,u^2 + (t - u) \text{ min. stress}\right)},$$

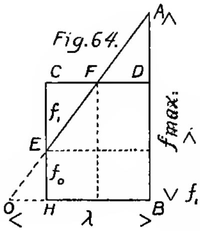
or from above,

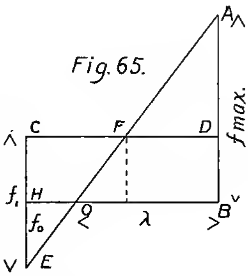
$$\text{Max. stress} = \tfrac{1}{2}\,u\left(1 + \sqrt{\left(1 + \frac{2 \text{ min. stress}}{u}\right)}\right).$$

The maximum stress or a must then be reduced by a reasonable amount to cover the uncertainties enumerated in § 167, so that $\frac{1}{2}$ a or $\frac{1}{3}$ a is taken for the maximum safe unit stress in designing. It also appears that for steady load or no variation, $a = t$; for a load from zero to u, $a = u = \frac{1}{2}\,t$; for a load from $+\,v$ to $-\,v$, $a = v = \frac{1}{3}\,t$. Experiments show more or less disagreement, as is to be expected, but the average results are as indicated.

163. **Fidler's Dynamic Theory.**—Fidler* suggests the following line of reasoning, as a *dynamic theory*, to account for the different *apparent* values of the unit stress which a material will endure when it is subjected to variable forces ranging between known limits.







As previously explained, a static load P_0 causing a unit stress B A, Fig. 63, will do work of elongation, (or compression) in increasing from zero to B A, equal to the area of the triangle O B A. A suddenly applied load P_1 which, if static, would cause a stress O C, since it does work equal to the product of O C into the elongation, must not be greater than one-half the static load, in order that O B $= \lambda$ and B A $= f$ max. may not be greater than before; for then area O B D C $=$ area O A B. The *real* stress in this latter case is twice the *apparent* stress found by dividing the load by the cross-section of the piece.

If the piece is already under a steady load stress H E, Fig. 64, which has caused the elongation (or compression) O H, the sudden imposition in addition of a load which, if

* Practical Treatise on Bridge Construction, T. C. Fidler; Philadelphia, Lippincott.

static, would cause the added stress E C, will do work equal to the rectangle E D and produce the final elongation O B = λ, since triangle E C F = triangle F D A. Then the real stress f max. $= f_0 + 2f_1$, or $= (P_0 + 2P_1) \div S$. Remember that f max. and λ are directly related.

Similarly, if $f_0 = \frac{1}{2} f$ max., and f_1 is due to a sudden increase of load, f_1 can only be one-half of f max. $- f_0$, if f max. and λ are to be kept within safe limits. Then, in general

$$\text{Actual stress} = f_0 + 2f_1 = (P_0 + 2P_1) \div S, \qquad (1.)$$

$$\text{Apparent stress} = (P_0 + P_1) \div S. \qquad (2.)$$

If P_0 is of the opposite sign to P_1, Fig. 65, the elongation from P_0 will be O H and the stress $- f_0$ = H E. The work done by P_1, if E C = D A = f_1, must be a rectangle equal to E C × C D = E C F + F D A, and f max. = A D + D B = A D + C E — H E.

$$\text{Actual stress} = 2f_1 - f_0 = (2P_1 - P_0) \div S.$$

$$\text{Apparent stress} = (P_1 - P_0) \div S.$$

The former expressions (1.) and (2.) are general, if the sign of P_0 is contained in the symbol. In the first case, $P_0 = 0$.

The stress will be equal to the load divided by the cross-section when the bar is in equilibrium, or at rest under a static load. When a part or all of the load is suddenly applied, the stress will momentarily be greater, then will be less, the bar will be thrown into vibration, and finally come to rest. When a load is applied rapidly but not suddenly, the maximum stress will be intermediate between that caused by a static and by a sudden loading.

On the above line of reasoning

$$\text{Equivalent static load} = P_0 + 2P_1 = P_0 + P_1 + P_1$$
$$= \text{Max. load} + (\text{Max.} - \text{Min.})\ \text{load},$$

or the maximum load of the strain sheet should be increased by the amount of the suddenly applied load, (or the max. stress — min. stress), and then used as a static load for the determination of the necessary cross-section. Members like stringers, cross-girders, and vertical suspenders in bridges, and

short girders should doubtless have this full allowance made. Those parts of bridges of considerable span, which do not receive full pull or thrust until a large portion of the span is loaded may have only one-half of this allowance added. See Pencoyd Bridge Co.'s allowance for impact, § 172.

Structures like roofs, subjected at long and rare intervals to the greatest loads for which they are computed, need very little if any allowance for variation of stress.

164. **Seefehlner's Rule.**—Seefehlner has proposed the following rule, founded on Wöhler's experiments:—

Let P = total maximum force, and P′ = total minimum force to which the bar is alternately subjected; n = ratio of P′ to P; u = rupturing or ultimate unit stress for a single application, as in a usual test; f = max. allowable unit stress, approaching the dangerous limit. Then

$$f = \tfrac{2}{3}\left(1 + \frac{P'}{2P}\right)u = \tfrac{2}{3}(1 \pm \tfrac{1}{2}n)u.$$

If P′ is of the opposite kind to P, their ratio will be negative; hence the alternative sign is given to n.

Examples.—If u = 50,000 lbs. per sq. in. and

P′ = 0	P = 60,000 lbs.;	$f = \frac{2}{3}(1 + 0)$ 50,000 = 33,300 lbs.
P′ = 24,000 lbs.,	P = 72,000 lbs.;	$f = \frac{2}{3}(1 + \frac{1}{6})$ 50,000 = 38,900 lbs.
P′ = − 72,000 lbs.,	P = 72,000 lbs.;	$f = \frac{2}{3}(1 - \frac{1}{2})$ 50,000 = 16,700 lbs.

165. **Gerber's Parabola.**—Suppose that the ranges of stress for unlimited repetition are known for any material and are plotted as ordinates and the corresponding minimum stresses as abscissas; the points so located will fall in a parabolic curve.

Let f max. and f min. = limits of stress.

D = f max. ∓ f min. = range of stress, — being used for the same kind, and + for different kinds.

f = statical breaking stress. C = constant for the material.

Then Gerber's equation is

$$f^2 = (f\text{ min.} + \tfrac{1}{2}D)^2 + C\,D.$$

If f is known, and also f min. and f max. for any one range of stress, for unlimited repetitions before breaking, C can be determined, and then the limits of stress for all conditions of loading can be calculated.

For curves, see Unwin, "Testing of Materials of Construction," p. 392.

166. **Bauschinger's Laws.**—(a). The yield point is always immediately raised to that load with which a bar has been stretched. Thus, if the diagram first obtained is A B D, Fig 1, the

next application of the same or greater force will give C D. During a following period of rest, moreover, the yield point rises above the greatest load with which the bar has been stretched, quite noticeably in one day, and continues to rise for weeks and months.

[A bar must then experience some readjustment of particles after strain, in the interval of rest.]

(*b.*) The true elastic limit (limit of proportionality of change of length to stress producing it), is lowered, often to zero, by stretching the bar beyond the yield point; so that the specimen, if tested again, immediately after stretching and removal of load, has no true elastic limit, or a remarkably low one. But in a following period of rest the elastic limit rises again; it reaches, after several days, the load with which the bar was stretched, and, after sufficient time, certainly after some years, it is raised even above that load.

The distinction in behavior between elastic limit and yield point is quite wide. Mr. C. A. Marshall* did not find that the elastic limit was lowered by the action of a stress exceeding the yield point, provided the highest load was allowed to act long enough to produce its full extension.

(*c.*) The true elastic limit is raised by pulling a bar with loads above the elastic limit but below the yield point, and continues to rise after the loadings have ceased, and it rises the more the higher the load. The elastic limit reaches a maximum when it has been raised to near the yield point, and, by the application of loads beyond the yield point, it is thrown back according to law (*b.*) above.

Examples of elastic limits raised above the original yield point, as determined from a duplicate specimen, are not wanting, and it is the rule that the yield point is considerably and unmistakably raised by repeated stresses. Hence—

(*d.*) By often repeated tensions, of magnitude greater than the elastic limit and less than the yield point, the original yield point may be raised.

(*e*) By straining in tension beyond the elastic limit, the limit for compression is notably lowered; that is brought *towards* zero; and, conversely, by similar straining in compression, the elastic limit for tension is lowered. The higher the stresses, the greater the change. When an elastic limit, so lowered, is raised again by stresses in like direction and is then overstepped, the elastic limit for the opposite stress falls at once to zero or almost to zero. A rest of three or four days, and even weeks, has but little, if any, influence on these processes. Mr. C. A. Marshall found, however, that the elastic limit in compression for small steel rounds was completely restored by a rest of eighteen months, after the bars had been stretched by a load equal to yield point. The bars

*Technic 1889, Engineering Society, University of Michigan.

showed even a higher elastic limit and yield point than could have been found originally, viz., each about 41,000 pounds per square inch.

(*f.*) By gradually raised stresses, alternating between tension and compression, the elastic limit for the opposite stress cannot be lowered until the stresses pass the original elastic limit.

(*g.*) When an elastic limit for stress of one sign has been lowered by a previous straining of the opposite sign, it can be raised again by gradually increasing stresses alternating between tension and compression, but only to a limit which lies considerably below the original elastic limit. The limit which it is possible thus to reach is thought to be the *limiting magnitude of equal and opposite loads which may be borne with safety.* [May it not be supposed that the material has by the treatment been relieved of the stresses set up in the process of manufacture and brought to a normal state?] A member will not be broken by repeated reverse stresses until it is loaded up to and beyond these new limits.

(*h.*) Rupture is not caused by from five millions to sixteen millions of repetitions of tensile stresses whose lower limit is zero and whose upper limit lies in the neighborhood of the true elastic limit.

(*i.*) By often repeated stresses between zero and an upper stress which lies near to, or more or less above, the original elastic limit, this limit is raised beyond, often far beyond, the upper limit of the stresses; and it is raised the higher, the greater the number of repetitions, without, however, the possibility of exceeding a certain height.

(*k.*) Repeated stresses between zero and an upper limit, such as may raise the original elastic limit above the higher stress, do not produce rupture. If, however, the upper limit lies so high that the elastic limit cannot be raised beyond it, then rupture must follow after a limited number of such stresses.

This statement is more general than the earlier one, that the limiting safe stress lies just below the yield point. Although the yield point may rise during an endurance test, and the elastic limit may rise even above the original yield point, yet if the load at the beginning of a test exceeds the *original* yield point, and is repeated continuously, rupture will finally occur. It however appears to be possible to begin with a rightly chosen load, and to gradually increase it as the repetitions proceed, keeping at all times below the rising yield point, without producing rupture. In most practical cases such treatment cannot be applied. The yield point is easily determined for most specimens of iron and steel, while the trial to raise the elastic limit is slow, difficult and expensive.

(*l.*) The ultimate tensile strength is not lessened by millions of repetitions of stress, but is, if anything, raised, when the test piece, after such repetitions, is broken by a quiet load.

(*m.*) Stresses repeated millions of times on iron and steel cause no alteration of structure.

The alteration of structure which is found at the surface of rupture of a bar broken by repeated stresses appears to be confined to the fractured surface itself, as has been directly proved by etching the fractured surfaces. At the minutest distance below the surface of fracture the original structure is again evident. Hence the above statement.

There is, however, a peculiar marking which appears on the surface of a polished specimen while being stretched beyond the yield point, which amounts to an indication of final failure, and shows that the specimen has passed the yield point.

This evidence of no alteration of structure confines the idea of the "fatigue of metals" to cases in which complete recovery from stretch never takes place upon removal of stress. Instead of metal being *fatigued*, or being brought nearer to rupture by working stresses well below the limit given in (*g.*), it is really improved in tenacity and elasticity, while the ductility is not notably decreased unless there are flaws.

Fractures by repeated loadings below the apparent yield point of the metal as a whole are always *detail breaks*. Continuity is destroyed first at a flaw, or an overstrained spot, and from that point the fracture spreads by very minute, successive encroachments, until it has so weakened the cross-section that large inequalities of stress, amounting to the same thing as a cross-breaking stress, § 137, are produced in the yet sound portion, and the piece breaks with a fracture of the remainder resembling that produced by nicking and breaking by bending.

When there is time for the metal to rest, the piece assumes its former state. If it be strained up to the elastic limit in tension and then be allowed to rest, it can be strained safely afterwards to the elastic limit in compression. When, however, the stresses repeatedly alternate between compression and tension, plus and minus, the *algebraic* difference must be taken in determining the maximum allowable or limiting stress, which is the same as taking the numerical sum of the thrust and pull, or the range of stress, to guide in determining the safe stress or necessary section. Experimental evidence seems to show that, for continued repetitions of stress, such sum or range cannot ever exceed with safety the difference between zero and the elastic limit in either direction. Periods of rest between the intermediate applications of reverse stresses change the rule and permit of higher stresses. So some structures and some pieces will endure safely higher stresses than others. The beginner should fix in his mind the fact that pieces and parts frequently subjected to rapid change of stress in magnitude and sign, must be exposed to far lower unit stresses, if the life of the piece is of consequence.

167. Reduction of Unit Stresses.—The safe working stress to be used for any material will depend upon several considerations:—Whether the structure is to be temporary or

permanent; whether the load is stationary or variable and moving; if moving, whether its application is accompanied by severe dynamic shock and perhaps pounding; whether the load assumed for calculation is the absolute maximum; whether such maximum is applied rarely or likely to occur frequently; whether the stresses obtained are exact or approximate; whether there are likely to be secondary stresses due to moments arising from changes of the assumed frame; what the importance of the piece is in the structure, and the possible damage that might be caused by its failure.

The allowable unit stresses of different kinds, and for greater or less change of load, will be further reduced to provide against:—Distribution of stress on any cross-section somewhat different from that assumed; variations in quality of material; imperfections of workmanship, causing unequal distribution of stress; scantness of dimension; corrosion, wear or other deterioration from lapse of time or neglect; lack of exactness of calculation.

The allowable unit stresses so determined will be but a small fraction of the ultimate or breaking strength of the material; and it is evident that the idea that it will require several times the allowable maximum working load to cause a structure to fail is seriously in error.

Over-confidence of the inexperienced designer in the correctness of his design may be checked by a study of this section.

168. **Load and Impact.**—The design should be completely carried out, both in the principal parts and in the details. The latter require the most careful study, that they may be at once effective, simple and practical.* All the exterior forces which may possibly act upon a structure should be considered, and due provision should be made for resisting them. The static load, the live load, pressure from wind and snow, vibration, shock and centrifugal force should be provided for, as should also deterioration from time, neglect or probable abuse. A truss over a machine shop may at some time be used for supporting a heavy weight by block and

* A portion of these paragraphs is extracted from a lecture by Mr. C. C. Schneider.

tackle, or a line of shafting may be added; a small surplus of material in the original design will then prove of value. Light, slender members in a bridge truss, while theoretically able to resist the load shown by the strain sheet, are of small value in time of accident. The tendency from year to year is towards heavier construction.

Secondary stresses, as they are called, are due—first, to the moments at intersections or *joints*, when the axes of the members coming together at a connection do not intersect at a common point; and second, to the moments set up at joints by the resistance to rotation experienced by the several parts when the frame or truss is deflected by a moving load. If symmetrical sections are used for the members, if the connecting rivets are symmetrically placed, and if the axes of the intersecting members meet at one point, secondary stresses will be much reduced.

All members of a structure should be of equal strength, and the connections should develop the full strength of the body of the members connected. The connections should be as direct as possible. When a live load is joined to a static load, the judgment of the designer, or of the one who prepares the specifications for the designer, must be exercised. A warehouse floor, to be loaded with a certain class of goods, has maximum stresses from a static load. The floor of a drill-room, ball-room or highway bridge receives maximum loading from a crowd of people, the possible density of which can be ascertained. But if these masses of people keep moving, and more particularly if they keep step, the effect of their weight will be increased by the vibrations resulting therefrom. This action is generally called *impact*.

In the case of a building, the floor-joists, receiving the impact directly, will be most affected; the girders which carry the joists will be less affected; and the columns which support the girders will receive a smaller percentage of the impact, the proportionate effect growing less as the number of stories below the given floor increases. In the absence of trustworthy data from which to determine this impact, it is left to the judgment of the engineer to increase the live load by a certain percentage, or to decrease the allowable unit-stress,

for each case, to provide for the effect, as will be seen in the values given later.

The stresses produced by impact in railroad bridges are even more uncertain and ambiguous. The assumption is made by some that the effect of impact upon the several members is dependent upon the length of loaded distance which produces the maximum live load stress in a member, and a certain percentage in accordance therewith is added to the live load stresses. Others use different allowable working stresses on different members, depending upon the rapidity with which the load may be applied, and the proportion which the live load on the member bears to its static load.

For economy, make designs which will simplify the shop-work, reduce the cost and ensure ease of fitting and erection. Avoid an excess of blacksmith work and much use of bent pieces.

169. **Working Stresses for Timber.**—Average safe unit stresses, in pounds per square inch, for the more common woods, when subjected to moving loads with considerable shock. The last five lines have sometimes been prescribed for railway bridges. For static loads add 50 per cent.

	Tension.		Compression.		Extreme fibre stress.	Shear along grain.
	With grain.	Across do.	With grain.	Across do.		
White oak	1,000	200	1,400	500	1,000	200
Southern long leaf or Georgia pine	1,200	60	1,600	350	1,200	150
Douglas or Oregon fir or pine	1,200	....	1,600	300	1,100	150
Northern or short leaf yellow pine	900	50	1,200	250	1,000	100
Norway pine	800	....	1 200	200	700	....
Spruce	800	50	1,200	200	700	100
White pine	700	50	1,100	200	700	100
California redwood	700	....	....	.. ..	800	....
White Oak	1,100	....	1,000	300	1,200	135
Long leaf Southern pine	1,000	....	1,000	250	1,200	100
Oregon pine or fir	1,400	....	900	190	1,200	150
Spruce	700	....	800	150	800	85
White pine	600	....	800	150	1,000	85

In order to be on the safe side, low working stresses are usually assumed. If the actual maximum unit stress which could possibly come upon a member could be determined, including the secondary stress produced by the deformation of the system, a unit stress of considerably greater intensity might be used.

170. **Working Column Formulas for Timber.**—Formulas for wooden columns and posts have remained without much change for a number of years. The most common formula for the mean unit stress, $P \div S$, is, for the ends

$$\text{Flat, } \frac{1{,}000}{1 + \frac{l^2}{250\,h^2}}; \quad \text{One pin, } \frac{1{,}000}{1 + \frac{l^2}{190\,h^2}}; \quad \text{Two pins, } \frac{1{,}000}{1 + \frac{l^2}{125\,h^2}};$$

in which l = length in inches between bearings, and h = width of member in inches in the direction of greatest liability to bend. As most timber struts are rectangular, h is more convenient for use than r, the radius of gyration. For different woods, insert in place of 1,000 the compression value from the preceding table.

For roofs, if liberal allowance has been made for snow and wind pressure, these stresses may be increased 50 per cent.

For railway bridges and structures exposed to similar loading, when some allowance for deterioration from exposure is desired, the following formulas may be used:—For yellow or Southern pine,

$$\text{Flat, } 860 - 7\frac{l}{h}; \quad \text{One pin, } 860 - 8\frac{l}{h}; \quad \text{Two pins, } 860 - 9\frac{l}{h};$$

and for white pine,

$$\text{Flat, } 540 - 6\frac{l}{h}; \quad \text{One pin, } 540 - 6\tfrac{1}{2}\frac{l}{h}; \quad \text{Two pins, } 540 - 7\frac{l}{h}.$$

For highway bridges the above column formulas may be increased 25 per cent.; for roofs, 50 per cent., if liberal allowance for loads has been made.

171. **Former Tension Specifications for Iron and Steel.**. Permissible tensile stresses have gradually been modified. At first little or no discrimination was made between the effects.

of dead and live load, and a unit stress of 10,000 lbs. per square inch was generally used. Later, the allowable unit stress was modified for different parts of a structure, as shown by the following allowed tensile stresses for railway bridges in pounds per square inch:—

On lateral bracing, 15,000; on bottom chords and main diagonals (forged eye bars), 10,000; on do. do. (plates and shapes), net section; on counter rods and long verticals (forged eye bars), on solid rolled beams used as cross floor beams and stringers, on bottom flange of riveted cross-girders and riveted longitudinal plate girders over 20 feet long (net section), 8,000; on bottom flange of riveted longitudinal plate girders under 20 feet long (net section), 7,000; on counter rods and long verticals (plates and shapes), net section, 6,500; on floor beam hangers, and other similar members liable to sudden loading (bar iron with forged ends), 6,000; do. do. (plates and shapes), net section, 5,000.

For spans exceeding 150 feet, the above allowed tension in bottom chords and main diagonals and the compression in top chord sections, § 174, might be increased by

$$\left(\frac{150 \times \text{stress from dead load}}{\text{stress from dead and live load}} - 50\right) \text{ per cent.}$$

The effect of frequency and rapidity of loading, the ratio of live load stress to dead load stress, and the difference between bar iron and shape iron, as well as possibly the more even distribution of stress in a bar than in a rolled shape, caused the variation in values for unit stresses. Steel was not then in use for bridges.

Waddell, in 1887, gave, for highway bridges, loaded to 100 pounds per square foot, and spans up to 150 feet, unit stresses in pounds per square inch:

	IRON.	STEEL.
Lower chord bars and end main diagonals (forged eye bars)	10,000	12,500
Lower chords (plates and shapes), net section Middle panel diagonals and counters, adjustable Hip verticals (forged eye bars)	8,000	11,000
Middle panel diagonals and counters (plates and shapes), net section Hip verticals (plates and shapes), net section	7,500	10,000
Flanges of rolled beams Flanges of built beams (net section)	10,000	14,000
Beam hangers, loops	7,000	....
Beam hangers, plates	6,000	9,000
Lateral and vibration rods	15,000	....

Values to be increased with increase of span.

Two other specifications, based on the range of stress from maximum to minimum, are cited in § 174, in connection with the specifications for compressive stress.

The specifications for railway bridges first referred to were somewhat modified later. They gave the following allowable tensile stresses on wrought iron, in pounds per square inch, and permitted the use of steel as below:—

Floor-beam hangers and similar members liable to sudden loading (plates or shapes), net section, 5,000; do. do. (bar iron with forged ends), 6,000; lateral bracing, 15,000; solid rolled beams, used as cross floor-beams and stringers, 8,000; bottom flanges of riveted cross-girders, net section, 8,000; bottom flanges of riveted longitudinal plate girders, *over* 20 ft. long, used as track stringers, net section, 8,000; do. do. *under* 20 ft. long, net section, 7,000; bottom chords, main diagonals, counters and long verticals (forged eyebars), 8,000 for live loads, 16,000 for dead loads; do. do. (plates and shapes), net section, 7,500 for live loads, 15,000 for dead loads.

Medium steel might be used for tension members, plate girders and rolled beams with 20 per cent. increase of above stresses, and soft steel in place of wrought iron for all riveted work, under certain shop restrictions.

Union Pacific Railway bridge specifications, 1895, read:—

For live load, iron in rolled bars and in flanges of girders, 10,000 lbs. per sq. inch; plates and shapes, 8,000 lbs.

Unit stresses for dead load, to be allowed at fifty per cent. greater than the above; or, add $\frac{2}{3}$ dead load stress to live load stress and use the sum as all live load in computing sections.

For steel of 62,000 lbs. minimum strength, unit stresses 15 per cent. greater than the above are allowed.

172. Impact.—Pencoyd Iron Works, railway bridge specifications.—The effect of impact and vibration shall be considered and added to the maximum stresses resulting from engine and train-loads. The effect of impact is to be determined by the following formula:—Impact to be added to live load stress $=$ calculated live-load stress $\left(\frac{300}{L + 300}\right)$; where $L =$ length of loaded distance in feet which produces the maximum stress in member. The unit stress then allowed per square inch (*i. e.*, reduced to a static basis) is on soft steel, 15,000 lbs.; on medium do., 17,000 lbs.

Prof. Mélan proposes the following formula for percentage of increase of live load on railway bridges to bring it to an equivalent static load. If L is the *span* in feet,

Percentage of increase = 14 + 2,600 ÷ (L + 33), which reduces practically to $33\frac{1}{3}\%$ for L = 100, and 25% for L = 200.

173. **Working Stresses: Tension.**—The following requirements are acceptable for railway bridges:—

	Wrought Iron.	Soft Steel.	Medium Steel.
Chords, ties, counters, and long suspenders,	$7,500\left(1 + \frac{\text{min. stress}}{\text{max. stress}}\right)$	8,600 (do.)	9,400 (do.)
Plates and shapes, and bottom flanges of built girders,	$6,700\left(1 + \frac{\text{min. stress}}{\text{max. stress}}\right)$	7,700 (do.)	8,400 (do.)
Hangers, through pinhole,	5,300	6,000	6,600
Lateral and cross-section rods, wind,	20,000	23,000	25,000
Do., centrifugal force,	10,000	11,500	12,500
Long'l rods in trestles,	15,000	17,000	18,500

For highway bridges, add 25%.

For roofs and iron buildings:—

	Iron.	Soft Steel.	Medium Steel.
Bars, main members,	15,000	18,000	21,000
Shapes, net section, and bottom flanges of rolled beams,	12,000	14,000	16,000
Lateral bars,	20,000	23,000	25,000
Lateral angles, net section,	15,000	18,000	21,000

174. **Accepted Column Formulas.**—Column formulas for iron and steel members have been much modified from time to time.

The earliest formula, on which those at present in use are founded, was Gordon's, which was derived from Hodgkinson's experiments. The symbol h denotes the least dimension of the cross-section, or the dimension measured in the direction in which lateral flexure is most probable. This formula for iron columns of rectangular section was

$$P = 36{,}000\ S \div \left(1 + \frac{l^2}{3{,}000\ h^2}\right),$$

the last term of which was changed by Rankine into the general form $l^2 \div 36{,}000 r^2$, as $h^2 = 12r^2$ for a rectangle. For other forms of cross-section the ratio of the least dimension h to r would change. The two values 36,000 must not be confused; the numerical identity is accidental.

Sometimes the ratio $l \div h$ was represented by H or some similar symbol, and the formula was $P = f\,S \div (1 + a\,H^2)$

where f varied from 42,500 to 36,000, and a from $1 \div 5,820$ to $1 \div 2,700$ for columns with fixed ends and different forms of cross-section.

These values were derived from columns tested to failure, and the allowable mean unit stress in compression was then obtained by dividing f by $4 + 0.05H$, thus making the allowed unit stress smaller as the column became more slender.

Cooper's specifications for 1884 gave

$$P = 8,000S \div \left(1 + \frac{l^2}{40,000r^2}\right)$$

for square ends, 40,000 being changed to 30,000 and 20,000 for one and two pin ends respectively. No member was to have an unsupported length of more than 45 times its least width.

Jos. M. Wilson in 1885 gave the permissible mean unit stress for wrought iron members in compression, $f \div \left(1 + \frac{l^2}{36,000r^2}\right)$ for both ends fixed, with a substitution of 24,000 and 18,000 for 36,000, for the cases of one and two pin ends. But f was a variable determined as follows:—For pieces subjected to compression only, permissible unit stress $f = 6,500\left(1 + \frac{\text{min. stress}}{\text{max. stress}}\right)$ for rolled iron, and for pieces subjected alternately to tension and compression $f = 6,500\left(1 - \frac{\text{max. stress of lesser kind}}{2 \text{ . max. stress of greater kind}}\right)$. The required section for tension was to be found by either of the last two formulas, after substituting 7,500 for double rolled bars, and 7,000 for plates and shapes, in place of 6,500.

C. C. Schneider, in 1886, in a design for the arched bridge over the Harlem river, New York, proposed to use for compression members, $f \div \left(1 + \frac{f}{8E} \cdot \frac{l^2}{r^2}\right)$, where

$$f = 10,000\left(1 + \frac{\text{min. stress}}{2 \text{ max. stress}}\right)$$

for wrought iron members subjected to one kind of stress only, and

$$f = 10,000\left(1 - \frac{\text{min. stress}}{2 \text{ max. stress}}\right)$$

for similar members subjected to alternate tension and compression, no regard being paid to the sign of the stress. In the case of steel use 12,000 in place of 10,000. For tension members use the last two formulas alone for f.

The Pencoyd Iron Works, in 1887, gave for wrought iron,

$$14{,}000 \div \left(1 + \frac{l^2}{15{,}000r^2}\right) \text{ and } 18{,}000 \div \left(1 + \frac{l^2}{10{,}000r^2}\right)$$

for steel, limiting a compression member to a length not exceeding 120 times its least radius of gyration. The above formulas unmodified would apply to stresses from steady load. The rolling load stresses were to be increased by adding a variable percentage obtained as follows:—The effects of impact and vibration shall be considered and added to the maximum stresses resulting from engine and train loads. The effect of impact is to be found by multiplying the calculated max. *live load* stress by $(0.7 + \frac{5}{L})$, where L = number of panels loaded multiplied by a panel length (or loaded distance to produce the live load stress in question).

Colorado Midland Railway Specifications.

Main posts, chords and struts.

Flat ends..........	$(7{,}500 - 25\frac{l}{r})\left(1 + \frac{\text{min. stress}}{\text{max. stress}}\right)$			
One pin end.....	"	30 "	"	"
Two pin ends.....	"	35 "	"	"

Lateral and other struts, subject to maximum stresses as calculated, $11{,}000 - 45\frac{l}{r}$.

J. A. L. Waddell, in 1887, for highway bridges under a live load of 100 lbs. per sq. ft., specified for truss members,

Flat ends	$9{,}000 - 30\frac{l}{r}$	for iron.	$12{,}000 - 45\frac{l}{r}$	for steel.
One pin	" 35 "		" 53 "	
Two pins	" 40 "		" 60 "	

Cooper's specifications for 1890—Railway bridges.

	Wrought iron.	Medium steel.		
Chord segments.	$8{,}000 - 30\frac{l}{r}$	$10{,}000 - 36\frac{l}{r}$	for live load stresses.	
	16,000 — 60 "	19,000 — 72 "	" dead "	"
All posts	7,000 — 40 "	8,500 — 55 "	" live "	"
	14,000 — 80 "	17,000 — 110 "	" dead "	"
	10,500 — 60 "	13,000 — 85 "	" wind	"

Lateral struts $9{,}000 - 50\frac{l}{r}$ for assumed initial stresses.

All compression pieces are restricted to an unsupported length of less than 45 times the least width. No distinction is made between pin and fixed ends. The mean unit stress is seen to be reduced for pieces whose loads are more rapidly or frequently imposed.

175. Working Stresses: Compression.—Cooper's specifications for 1896 omit all requirements for wrought iron, and require for medium steel—

Chord Segments	$10,000 - 45\frac{l}{r}$	for live load stresses.
	$20,000 - 90$ "	" dead " "
Posts of Through Bridges	$8,500 - 45\frac{l}{r}$	" live " "
	$17,000 - 90$ "	" dead " "
Posts of Deck Bridges and Trestles	$9,000 - 40\frac{l}{r}$	" live " "
	$18,000 - 80$ "	" dead " "
Lateral Struts and Rigid Bracing	$13,000 - 70\frac{l}{r}$	
	For live load, two-thirds of same.	

Soft steel may be used in compression, with unit stresses 15 per cent. less than those allowed for medium steel.

A recent specification for railway bridges gives:

	Wrought Iron.	Soft Steel.	Medium Steel.
Flat ends.........	$8,600 - 27\frac{l}{r}$	$10,000 - 33\frac{l}{r}$	$11,000 - 38\frac{l}{r}$
One pin..........	" 30 "	" 37 "	" 43 "
Two pins.........	" 33 "	" 41 "	" 48 "

For highway bridges, increase 25 per cent.

For roof trusses and buildings the mean unit compressive stress may be taken as, for

	Wrought Iron.	Soft Steel.	Medium Steel.
Flat ends.........	$10,750 - 33\frac{l}{r}$	$12,500 - 42\frac{l}{r}$	$13,750 - 48\frac{l}{r}$,

which values are the same as indicated above for highway bridges.

For cast-iron, f may be taken as 10,000, and a as $1 \div 60,000$. The use of cast-iron for compression members and beams is not approved.

The compression flange of built beams and girders is usually made of the same *gross* section as that of the tension flange.

176. Alternating Stresses.—Good practice requires a larger section to be given to members which are alternately subjected to tension and compression than would be needed if the stress were always of one sign. The discussion in §§ 156–8 makes the reason plain.

Here, as well as in the specifications for columns, there has been much change, and the present tendency is to adopt the Launhardt-Weyrauch formula or an analogous one.

An old specification for wrought iron members was—Required section = max. tension ÷ 10,000 + max. compression ÷ $\frac{1}{4}$ column strength, in case this result is larger than that given by the usual formula for columns. The phrase one-fourth column strength was intended to mean one-fourth of the unit stress given by the column formula when f was taken as 36,000 to 40,000, § 174.

The Pencoyd specifications say—Members subjected to alternate stresses of tension and compression shall be so proportioned that the total sectional area is equal to the sum of the areas required for each stress.

Theodore Cooper specifies—Members subject to alternate stresses of tension and compression shall be proportioned to resist each kind of stress. Both of the stresses shall, however, be considered as increased by an amount equal to 0.8 of the least of the two stresses, for determining the sectional areas.

Another engineer says—For compressive stress alone, use formula for posts. For the greater kind of stress, use

$$7{,}000\left(1 - \frac{\text{max. lesser stress}}{2\ \text{max. greater stress}}\right).$$

J. A. L. Waddell prescribes—In any portion of a bridge in which the stresses of tension and compression alternate, the sectional area required is to be determined by dealing with the part thus affected,—first for the calculated maximum tension, then for the calculated maximum compression,—employing for unit stresses the values that would be used were there no reversion of stress $\times \left(1 - \frac{1}{2}\,\frac{\text{smaller stress}}{\text{larger stress}}\right)$, and adopting the greater of the two areas thus found.

A recent specification for railroad bridges gives, for the greater stress, a value

Wrought Iron.	Soft Steel.	Medium Steel.
$7{,}500\left(1 - \frac{\text{max. lesser stress}}{2\ \text{max. greater stress}}\right)$	8,600 (1 — do.)	9,400 (1 — do.)

and for highway bridges

9,400 (1 — do.)	10,800 (1 — do.)	11,700 (1 — do.)

or 25 per cent. more.

In specifications for the Hudson River suspension bridge, Mr. Cooper gives, for the stiffening trusses—Under the reversal of stress by live loads, the chords of the stiffening trusses shall not be subjected to a greater unit stress per square inch than

$$\text{unit tension} = \frac{T}{T + a\,C}\,20{,}000,$$

$$\text{unit compression} = \frac{a\,C}{T + a\,C}\left(20{,}000 - 70\,\frac{l}{r}\right);$$

nor the web diagonals of the stiffening trusses to a greater unit stress than

$$\text{unit tension} = \frac{T}{T + a\,C}\,18{,}000,$$

$$\text{unit compression} = \frac{a\,C}{T + a\,C}\left(17{,}500 - 75\,\frac{l}{r}\right);$$

where the quantity in parenthesis is the column formula; T = total tension in the member; C = total compression in the member; and a = value of the net section in terms of the gross section of the member.

177. **Shearing and Bearing Stresses.**—Working unit stresses in pounds per square inch.

		Wrought Iron.	Soft Steel.	Medium Steel.
Shear on pins and rivets	Railway bridges	7,000	8,000	8,700
	Highway bridges	8,000	9,000	10,000
	Roofs and buildings	9,000	10,000	11,000
Shear on webs of girders	Railway bridges	5.000	5,700	6,800
	Highway bridges	5,500	6,500	7,500
	Roofs and buildings	6,000	7,000	7,500
Bearing on diameter of pins and rivet holes	Railway bridges	12,000	14,000	15,000
	Highway bridges	15,000	17,000	19,000
	Roofs and buildings	18,000	20,000	22,000
Bending stress on pins.	Railway bridges	15,000	17,000	19,000
	Highway bridges	17,500	20,000	22,000
	Roofs and buildings	20,000	22,500	25,000
Bending stress on Purlins		12,000	14,000	16,000

The number of field rivets should be 25 per cent. in excess ot that required for power riveting.

For lateral connections 25 per cent. greater stress may be allowed.

For eyebar heads the stress in bearing may be 25 per cent. greater than that in the bar.

178. **Pedestals.**—For bearing plates and pedestals, the working compression on blocks per square foot, in pounds, may be, for granite, 95,000; limestone, 80,000; sandstone, 45,000; brickwork, 6,700; concrete, 7,000.

For rollers, not less than 2 in. in diameter, the pressure in pounds per linear inch of roller shall not exceed $700\sqrt{d}$ for wrought iron, or $900\sqrt{d}$ for steel, d being taken in inches.

The pressures transmitted to the masonry by the pedestals shall not exceed 40,000 pounds per square foot under maximum possible loading; and pressure within the masonry or upon rock foundation shall not exceed 20,000 pounds per square foot.

The student is advised to study carefully two or more standard specifications for bridge or structural work, and to note the requirements for members and their details. The discussions in this book embody an attempt to make clear the reasons for such requirements.

CHAPTER XI.

INTERNAL STRESS: CHANGE OF FORM.

179. **Introduction.**—Let any body to which forces are applied be cut by a plane of section. Stresses of tension, compression or shear,—normal, oblique or tangential,—may exist between the particles at the section. It is desirable to know the magnitude and kind of the unit stress at each point in order to be sure that the material can safely resist it; or to determine the required cross-section to reduce the existing stresses to safe values.

A unit stress is expressed as a certain number of pounds of tension, compression, or shear on a square inch of section. If the plane of section is changed in direction, the force on the section may be changed and the area of section may also be changed, so that the unit stress on the new section is altered from that on the old in two ways. Stresses per square inch, or unit stresses, therefore cannot be resolved and compounded as can forces, unless they happen to act on the same plane. Generally, each unit stress may be multiplied by the area over which it acts, and the several *forces* so obtained may be compounded or resolved as desired; the final force or forces divided by the areas on which they act will give the desired unit stresses.

Where the stress on a plane varies from point to point, as does the direct stress on the right section of the beam, and as does the shearing stress also in the same case, the investigation is supposed to be confined to so small a portion of the body that the stress over any plane may be considered to be sensibly constant.

180. **Stress on a Section Oblique to a Given Force.**—Suppose a short column or bar to carry a force of direct compression or tension, of magnitude P, centrally applied and uniformly distributed over the cross-section S. The unit stress on and perpendicular to the right section will be $p_1 = P \div S$.

On an oblique section C D, Fig. 2, making an angle θ with the right section A B, the unit stress will be $P \div S \sec \theta = p_1 \cos \theta$, making an angle of θ with the normal to the oblique section on which it acts. If this oblique unit stress is resolved normally and tangentially to C D, the

$$\text{Normal unit stress} = p_n = p_1 \cos \theta \,.\, \cos \theta = p_1 \cos^2 \theta;$$

$$\text{Tangential do.} = q = p_1 \cos \theta \sin \theta.$$

The normal unit stress on the oblique plane is of the same kind as P, tension or compression; the tangential unit stress, or shear, tends to make one part of the prism slide down and the other part slide up the plane.

The largest normal unit stress for different planes is found when $\theta = 0$, which defines the fracturing plane for tension; the minimum normal unit stress occurs for $\theta = 90°$; and the greatest unit shear is found for $\theta = 45°$, when we have q max. $= \frac{1}{2} p_1$.

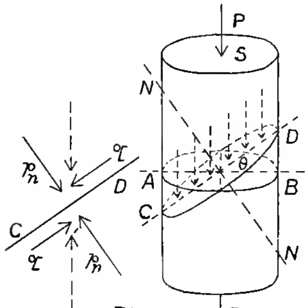


Fig. 2.

181. Combined Stresses.—The action line of P may be taken for the axis of X. Two equal and opposite forces, pull or thrust, may then be applied along the axis of Y, and the normal and tangential unit stresses found on the plane just discussed; and similarly for the direction *Z*. The normal unit stresses, since they act on the same area, may then be added algebraically, and the shearing stresses may be combined; finally a resultant oblique unit stress may be found on the given plane.

A more convenient method will, however, be developed and used in the following sections. As most of the forces which act on engineering structures lie in one plane or parallel planes, such cases chiefly will be considered.

182. Unit Shears on Planes at Right Angles.—If, in the preceding illustration, the unit stresses, both normal and tangential, are found on another plane N which makes an angle $\theta' = 90° - \theta$ with the right section, there will result

$$p'_n = p_1 \cos^2 \theta' = p_1 \sin^2 \theta; \quad q' = q.$$

Hence, on a pair of planes of section at right angles to one another the *unit shears are of equal magnitude*, and, since $p_n + p'_n = p_1$, the unit normal stresses are together equal to the original normal unit stress. It is further evident that one normal unit stress p'_n may be found by subtraction as soon as the other is known, and that ordinary resolution on these two planes of the original unit stress would be erroneous.

183. Unit Shears on Planes at Right Angles Always Equal.—Since, as before stated, other forces, in other directions, may be simultaneously applied to the given body, and their effects found on the same two planes, it follows that, in any body under stress, the *unit shearing stress, on each of any two planes at right angles, will be equal*:—a very valuable principle.

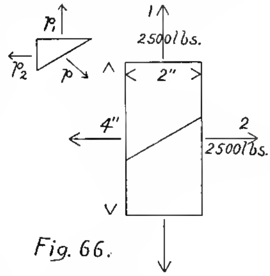


Fig. 66.

Example.—A closed cylindrical receiver, $\frac{1}{4}$ in. thick, has a spiral riveted joint making an angle of $30°$ with the axis of the cylinder, and a portion 2 in. $\times$ 4 in. of the cylinder, Fig. 66, has the given tensions of 2,500 lbs. acting upon it. Then $p_1 = 2{,}500 \div 2 \quad \frac{1}{4} = 5{,}000$ lbs. per sq. in., and $p_2 = 2{,}500 \div 4 \quad \frac{1}{4} = 2{,}500$ lbs. per sq. in.

$$p_n = 5{,}000 \cdot \tfrac{3}{4} + 2{,}500 \cdot \tfrac{1}{4} = 4{,}375 \text{ lbs. per sq. in.}$$
$$q = 5{,}000 \quad 0.433 - 2{,}500 \cdot 0.433 = 1{,}082 \text{ lbs. per sq. in.}$$
$$p = \sqrt{(4{,}375^2 + 1{,}082^2)} = 4{,}507 \text{ lbs. per sq. inch.},$$

or $4{,}507 \quad \frac{1}{4} = 1{,}127$ lbs. per linear inch of joint, which value will determine the necessary pitch of the rivets for strength.

The stress on a joint at right angles to the above can be similarly found. An easier process will be given in § 190.

184. Compound Stress is the internal state of stress in a body caused by the combined action of two or more simple stresses (or balanced sets of external forces) in different directions, as in the above example. The investigations which follow are those of compound stress, but they will, as above stated, be chiefly confined to stresses in or parallel to one plane.

185. Conjugate Stresses: Principal Stresses.—If the stress on a given plane in a body is in a given direction, the stress on any plane *parallel* to that direction must be parallel to the first-mentioned plane. For the equal resultant forces

exerted by the other parts of the body on the faces A B and C D of the prismatic particle, Fig. 67, are *directly opposed* to one another, their common line of action traversing the axis of X through O; and they are therefore independently balanced. Therefore the forces exerted by the other parts of the body on the faces A D and B C of this prism must be independently balanced and have their resultants directly opposed; which cannot be unless their direction is parallel to the plane Y O Y.

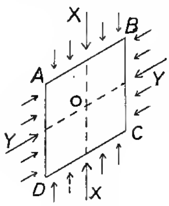


Fig. 67.

A pair of stresses, each acting on a plane parallel to the direction of the other, is said to be *conjugate*. Their unit values are independent of each other, and they may be of the same or opposite kinds. If they are normal to their planes, and hence at right angles to each other, they are called *principal* stresses.

Examples.—The unit stress found in § 183 makes an angle with the plane on which it acts whose tangent is 4,375 ÷ 1,082 = 4.04. Upon a new plane cutting the metal in this direction the stress must act in a direction parallel to the joint referred to.

If a plane be conceived parallel to a side-hill surface, at a given vertical distance below the same, the pressure at all points of that plane, being due to the weight of the prism of earth above any square foot of the plane, will be vertical and uniform. Then must the pressure on a vertical plane transverse to the slope be parallel to the surface of the ground. That the pressure against the vertical plane is not horizontal, but inclined in the direction stated, is shown by the movement of sewer trench sheeting and braces, when the braces are not inclined up hill, but are put in horizontally.

186. Shearing Stress.—If the stresses on a pair of planes are *entirely* tangential to those planes, the unit shears must be equal. Consider them as acting along the faces of a small prismatic particle A B C D, which lies at O. The moment of the total shear on the two faces A B and C D must balance the moment for the faces A D and B C, for equilibrium.

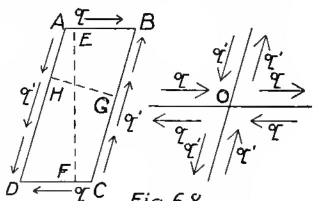


Fig. 68.

$$q' \,.\, AB \,.\, EF = q \,.\, AD \,.\, HG \,.$$

But the area of A B C D, A B . E F = A D . H G; $\therefore\ q' = q$.

This construction shows further that a shear cannot act alone as a simple stress, but must be combined with an equal unit shear on a different plane.

187. Two Equal and Like Principal Stresses.—If a pair of *principal* stresses, § 185, are equal unit stresses of the same kind, p_1 and p_2, Fig. 69, the stress on *every* plane is *normal* to that plane, and of the same kind and magnitude.

Let p_1 act in the direction O X on the plane O′ B′ of the prismatic particle O′ A′ B′ which lies at O, and p_2 act in the direction O Y on the plane O′ A′, p_1 and p_2 being equal unit stresses of the same kind. Make O D $= p_1$. O′ B′, the total force on O′ B′, and O E $= p_2$. O′ A′, the total force on O′ A′, both being positive. Complete the rectangle O D R E. Then must R O, applied to the plane A′ B′, be necessary to insure equilibrium of the prism O′ A′ B′. Hence $p' =$ R O $\div$ A′ B′ Since $p_1 = p_2$,

$$\frac{\mathrm{O\ D}}{\mathrm{O'\ B'}} = \frac{\mathrm{O\ E}}{\mathrm{O'\ A'}} = \frac{\mathrm{O\ R}}{\mathrm{A'\ B'}}\,; \quad \therefore p' = p_1 = p_2 .$$

Because of similarity of triangles A′ O′ B′ and O E R, R O is

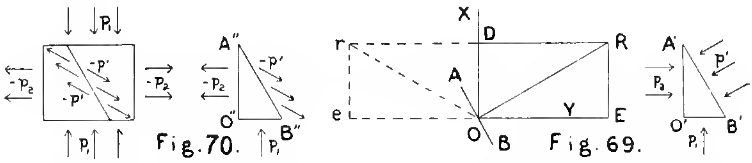


Fig. 70. Fig. 69.

perpendicular to A B, or p' to A′ B′, and is of the same kind as p_1 and p_2.

Example.—Fluid pressure is normal to every plane passing through a given point, and equal to the pressure per square inch on the horizontal plane traversing the point. Here manifestly the three co-ordinate axes of X, Y and Z might be taken in any position, as all stresses are principal stresses.

188. Two Equal and Unlike Principal Stresses.—If a pair of *principal* stresses are equal unit stresses of opposite kinds, as p_1 and $-p_2$, Fig. 70, the unit stress on every plane will be the same in magnitude, but the angle which its direction makes with the normal to its plane will be bisected

by the axis of principal stress, and its kind will agree with that of the principal stress to which it lies the nearer.

In this case lay off Oe in the negative direction, to represent $-p_2$ O″ A″; construct the rectangle O Dre, and draw rO which will be the required force distributed over A″ B″ to balance the forces O″ B″ and O″ A″. This force rO will be the same in magnitude as R O, making $p' = p_1 = p_2$ and rO will make the same angle with O X or O Y as R O does. As R O lies on the normal to A B, and O X bisects R Or, the statement as to position is proved. The stress p' agrees in kind with that one of the principal stresses to which its direction is nearer.

189. Two Shears at Right Angles Equivalent to an Equal Pull and Thrust.—If the plane A″ B″ is at an angle of 45° with O X, rO will coincide with A B and becomes a shear. Therefore two equal unit stresses of opposite kinds, that is a pull and a thrust, and normal to planes at right angles to one another, are equivalent to, and give rise to, equal unit shears on planes making 45° with the first planes and hence at right angles to each other; and *vice versa*.

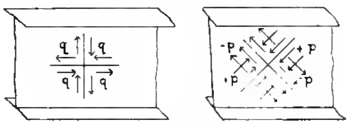


Fig 71.

Example.—If, at a point in the web of a plate girder, Fig. 71, there is a unit shear, and nothing but shear, on a vertical plane, of 4,000 lbs., there must be a unit shear of 4,000 lbs., and nothing but shear, on the horizontal plane at that point; and on the two planes inclined at 45° to the vertical through the same point there will be, on one, only a normal unit tension of 4,000 lbs., and on the other an equal normal unit thrust.

From a combination of the two preceding demonstrations follows the more general problem.

190. Stress on any Plane, when the Principal Stresses are Given.—Let the two principal unit stresses be $p_1 =$ O D, and $p_2 =$ O F, of any magnitude, and of the same kind or opposite kinds. Fig. 72. The direction, magnitude and kind of the unit stress on any plane A B is desired.

Let p_1 be the greater. Divide p_1 and p_2 into their half sum and difference as follows:—

$$p_1 = \tfrac{1}{2}(p_1 + p_2) + \tfrac{1}{2}(p_1 - p_2), \text{ and } p_2 = \tfrac{1}{2}(p_1 + p_2) - \tfrac{1}{2}(p_1 - p_2).$$

The distance O C or O E will represent the half sum $\frac{1}{2}(p_1 + p_2)$, and C D or E F the half difference $\frac{1}{2}(p_1 - p_2)$. If p_1 and p_2 are of the same sign the right hand figure will result; if of opposite signs, the left hand figure will be obtained.

By the principle of § 187, when the two equal principal unit stresses O C and O E are considered, lay off O M on the normal to the plane whose trace is A B, for the direction and magnitude of the unit stress on A B due to

$\frac{1}{2}(p_1 + p_2)$.

There remain C D and E F representing $+ \frac{1}{2}(p_1 - p_2)$ on the vertical axis, and $- \frac{1}{2}(p_1 - p_2)$ on the horizontal axis respectively.

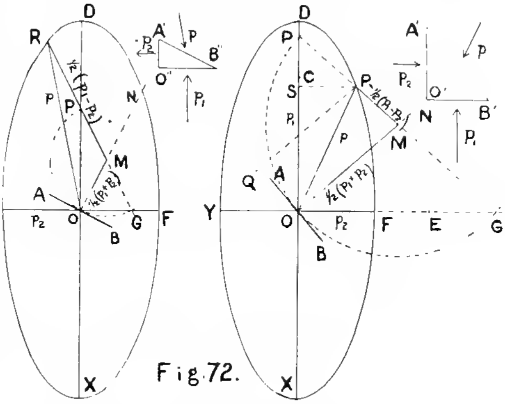


Fig. 72.

By § 188, lay off O Q, making the same angle with O X as does O M, but on the opposite side, or in the contrary direction, for the magnitude and direction of stress on plane A B due to $\pm \frac{1}{2}(p_1 - p_2)$. As O M and O Q both act on the same unit of area of A B, R O, in the opposite direction to their resultant O R, will give the direction and magnitude of the unit stress on A′ B′ to balance p_1 on O′ B′ and p_2 on O′ A′. In the figure on the right R O is positive, or compression. If, in the figure on the left, where p_1 is + and p_2 is —, R O falls so far to the right as to come on the other side of A B, it will agree with p_2, and be negative or tension. If A B is taken much more steeply inclined, such will be the case. The small prisms illustrate the constructions. If R O falls on A B, it will be shear. Some constructions for different inclinations of plane A B will be helpful to an understanding of the matter.

A much abbreviated construction is as follows:—Strike a semicircle from M on the normal, with a radius $M O = \frac{1}{2}(p_1 + p_2)$, and draw M R through the points where the semicircle cuts

the axes of p_1 and p_2. The angle N M R is thus double the angle M O D, since it is an exterior angle at the vertex of an isosceles triangle. Lay off M R $= \frac{1}{2}(p_1 - p_2)$ in the direction of the axis of greatest stress, and R O will be the desired unit stress on A B. If p_2 is opposite in kind to p_1, M R will be greater than M O, and R will go beyond P.

191. Ellipse of Stress.—For different planes A B through O, p_1 and p_2 being given, the locus of M is a circle of radius $\frac{1}{2}(p_1 + p_2)$, and the locus of R is an ellipse (as will be proved below), with major and minor semidiameters p_1 and p_2. Hence it is seen why p_1 and p_2, normal to the respective planes, and at right angles, are called *principal* stresses.

If three principal stresses, coinciding in direction with the rectangular axes of X, Y and Z, simultaneously act on a given point, an ellipsoid constructed on them as semidiameters will limit and determine all possible stresses on the various planes which can be passed through that point in the body.

That the locus of R is an ellipse may be proved as follows:—Drop a perpendicular R S from R on O X. P M O and O M G are isosceles triangles. $\angle \text{POM} = \angle \text{GOB} = \theta$.

$$\text{OM} = \text{MP} = \tfrac{1}{2}(p_1 + p_2);\ \ \text{GR} = p_1;\ \ \text{PR} = p_2.$$
$$\text{RS} = \text{PR} \sin \text{MPO} = p_2 \sin \text{POM} = p_2 \sin \theta,$$
$$\text{SO} = \text{GR} \sin \text{MGO} = p_1 \cos \text{POM} = p_1 \cos \theta,$$
$$\text{OR} = p_r = \sqrt{(\text{SO}^2 + \text{RS}^2)} = \sqrt{(p_1^2 \cos^2 \theta + p_2^2 \sin^2 \theta)}. \quad (1.)$$

which is the value of the radius vector of an ellipse, the origin being at the centre.

$$\text{As } \angle \text{NMR} = 2 . \angle \text{POM} = 2\,\theta,$$
$$\sin \text{NOR} : \sin \text{OMR} = \text{RM} : \text{OR} = \tfrac{1}{2}(p_1 - p_2) . p_r;$$
$$\therefore \quad \sin \text{NOR} = \sin 2\,\theta \,.\, \frac{p_1 - p_2}{2p_r}, \quad (2.)$$

which gives the obliquity of the unit stress to the normal to the plane, in terms of the angle of the normal with the axis of greater principal stress, or of the plane with the other axis. The graphical construction gives the stress and its angle with the normal or the plane by direct measurement, and is far more convenient and less liable to error.

If $p_1 = p_2$, the case reduces to that of § 187 or 188.

If the ellipse whose principal semi-diameters are p_1 and p_2 is given, the unit stress on any plane may briefly be found by drawing the *normal* to the plane, laying off O M $= \frac{1}{2}(p_1 + p_2)$, taking a radius of $\frac{1}{2}(p_1 - p_2)$, and, with M as a centre, cutting the ellipse at R on the side of the normal towards the greater axis. A line R O will be the desired unit stress.

Example.—Let $p_1 = 100$ lbs. on sq. in., $p_2 = -50$ lbs. Plane A B makes 30° with direction of p_2, or its normal makes 30° with p_1. Construct the figure and find the magnitude, direction and sign of the unit stress on the oblique plane. Try other values.

192. Shearing Planes.—To determine the angle of the planes on which there is only shear, and the conditions which render shearing planes possible.

If the plane A B of the previous figure is to be the shearing plane, there must be no normal component upon it, and therefore, from § 180, if the plane makes an angle θ_s with p_2, or the normal to it is inclined at an angle θ_s to p_1,

$$p_n = p_1 \cos^2 \theta_s + p_2 \sin^2 \theta_s = 0.$$

$$\therefore \frac{\sin \theta_s}{\cos \theta_s} = \tan \theta_s = \sqrt{\left(\frac{-p_1}{p_2}\right)}.$$

No shearing plane is possible unless p_1 and p_2 differ in sign. There will then be two planes of shear making equal angles with the direction of p_2 or of p_1.

In the above example, $\sqrt{(100 \div 50)} = \sqrt{2} = \tan \theta_s = 1.4142$. $\theta = 54° \, 44'$.

If the ellipse of stress is drawn, take a radius equal to the side of a right angled triangle whose other side is $\frac{1}{2}(p_1 + p_2)$, and hypothenuse is $\frac{1}{2}(p_1 - p_2)$, and strike a circle from the centre of the ellipse. Planes drawn through the points of intersection of this circle and ellipse and the centre will be the shearing planes. Unless p_1 and p_2 differ in sign, the circle will be imaginary. The value of the shear on these planes is

$$q = \sqrt{\left(\tfrac{1}{4}(p_1 - p_2)^2 - \tfrac{1}{4}(p_1 + p_2)^2\right)} = \sqrt{p_1 p_2}.$$

193. Given any two Stresses: to find Principal Stresses.– As, in actual practice, two oblique unit stresses on different planes may often be known in magnitude, direction and sign, it will then be required to find the principal unit

stresses, since one of them is the maximum stress to be found on any plane, and the other is the minimum stress of the same kind, or the maximum normal stress of the opposite kind.

Given two existing unit stresses, p and p', of any direction, magnitude and sign, to determine the principal unit stresses, p_1 and p_2.

If p_1 and p_2 were known, and p and p' were then to be found from the former, the construction shown in Fig. 73 would be made, in which O M $=$ O M′ $= \frac{1}{2}(p_1 + p_2)$ and M R $=$ M′ R′ $= \frac{1}{2}(p_1 - p_2)$. If one of these normals were revolved around O to coincide with the other, the point M′ would fall on M, but M′ R′ would diverge from M R, while equal in length to it.

Hence, when p and p' are the given quantities, let A B, Fig. 74, represent the trace of the plane on which p acts, O N the normal to that plane, and O R the unit stress p in magnitude and actual direction of action on A B. O R represents either tension or compression, as the case may be. Now let the plane on which p' acts, together with its normal and p' itself in its relative position, be revolved about O until it coincides with A B. Its normal will fall on O N and p' will be found at O R′, on one side or the other of O N, if it is of the same kind with p; or it is to be laid off on one of the dotted lines below, if of the opposite sign.

In other words, lay off p' from O, at the same angle with O N which it makes with the normal to its own plane. It is well, for accuracy of construction, to draw it on the same side of the normal as p, the result being the same as if it were drawn on the other side. (The change from one side of the normal to the other simply consists in using the corresponding line on the other side of the main axis of the ellipse of stress). Thus is found O R′ or — O R′ as the case may be. Draw R R′ and, since R M R′ must be an isosceles triangle, bisect R R′ at T and drop a perpendicular to R R′ from T on to the normal, cutting the latter at M. Then since, as previously pointed out, O M $= \frac{1}{2}(p_1 + p_2)$ and M R $=$ M R′ $= \frac{1}{2}(p_1 - p_2)$, — with M as a centre, and radius M R, describe a semicircle; O N will be p_1 and O S will be p_2. Since p is in its true position, and the angle N M R $=$ 2 M O D of Fig. 72 or

2 M O X of Fig. 73, the direction of the axis X along which p_1 acts will bisect N M R, and the axis along which p_2 acts will be perpendicular to axis X. They may be now at once drawn through O, if desired.

194. From any two Stresses to find other Stresses.—From the preceding construction, § 193, the stress on any other plane may now be found. All *possible* values of p consistent with the two, O R and O R′, first given, will terminate, in Fig. 74, on the semi-circle just drawn, as at R″, and the greatest possible obliquity to the normal to any plane through O will be found by drawing from O a tangent to this semi-circle.

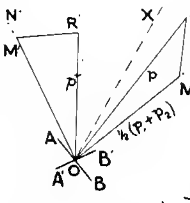


Fig. 73

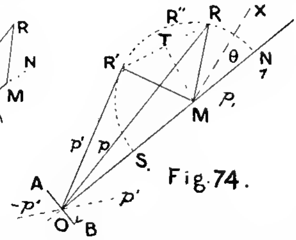


Fig. 74.

195. When Shearing Planes are Possible.—In case the lower end of the semi-circle cuts below O, Fig. 75, p_1 and p_2 are of opposite signs, all obliquities of stress are possible, and the distance from O to the point where the semi-circle cuts A B, being perpendicular to the normal O N, gives the unit shear on the shearing planes. If p_1 and p_2 are drawn through O in position, and the ellipse of stress is then constructed on them as semi-diameters, (as can be readily done by drawing two concentric circles with p_1 and p_2 respectively as radii, and projecting at right angles, parallel to p_1 and p_2, to an intersection, the two points where any radius cuts both circles), an arc described from O, with a radius equal to this unit shear and cutting the ellipse, will locate a point in the shearing plane which may then be drawn through that point and O. Two shearing planes are thus given, as was proved to be necessary, § 186.

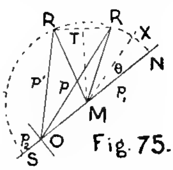


Fig. 75.

The above solution may be considered the general case.

196. From Conjugate Stresses to find Principal Stresses.—If p and p' are conjugate stresses, it is evident, from definition, and from Fig. 67, that they are equally

inclined to their respective normals. Hence O R′ will fall on O R, when revolved, both O R and O R′ lying above O when of the same sign, and on opposite sides of A B when of opposite signs. The rest of the construction follows as before, being somewhat simplified. When one conjugate stress is shear, the other is shear, of the same unit value, by § 186, but the stress on any other plane cannot be shear.

It may be noted that, when $p_1 = \pm p_2$, the propositions of § § 187 and 188 are again illustrated.

One who is interested in a mathematical discussion of this subject is referred to Rankine's "Applied Mechanics," where it is treated at considerable length. This graphical discussion is much simpler, less liable to error, and determines the stresses in their true places.

For the application to the pressure of earth against a wall or any plane see § 255.

197. Change of Principal Stresses by a Shear.—The varying tensile and compressive stresses on any section of a

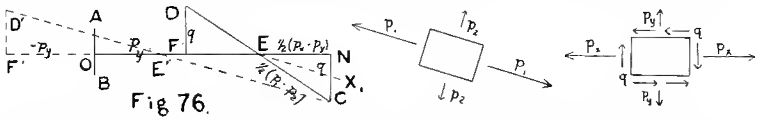


Fig 76.

beam are accompanied by varying shearing stresses on that section and by equal shears on a longitudinal section. The direct or normal stresses due to bending moment vary uniformly from the neutral axis either way, § 75; the shears are most intense at the neutral axis, § 86. The maximum and minimum stresses on any particle with their directions may be found as follows:—

Given the normal and tangential components of the stresses on two planes at right angles, to find the magnitude and direction of the new principal stresses. This problem is a special case of § 193, p and p' being given by their components. Let the original normal unit stress, due to the bending moment, and normal to the section A B, Fig. 76, be denoted by p_x. If there is any normal stress on the plane at right angles to A B, denote it by p_y. See § § 198, 206. The two equal unit shears on the two faces will be q.

Lay off $p_x =$ O N on the normal to the plane A B on which it acts in its true direction. As q acts on the same area

of A B as does p_x, lay off N C $= q$ at right angles. The two component stresses being O N and N C, a line from O to C will represent the direction and magnitude of the unit stress on A B. Revolve the plane on which p_y acts through 90°, to coincide with A B, so that the normals coincide. Then will p_y fall at O F, if of the same sign as p_x, or at O F′ if of the opposite sign. F D or F′ D′ is q, the revolved shear, laid off in the direction opposite to that of N C, as its direction requires,—see sketch on the right. Then a line from O to D or D′ will be the revolved stress. As O C and O D represent p and p', by § 193, connect D with C; bisect D C at E, which point falls on O N and is also the point where the perpendicular from D C will strike O N. Hence

$$\text{O E} = \tfrac{1}{2}(p_x + p_y) = \tfrac{1}{2}(p_1 + p_2);$$

$$\text{E C} = \tfrac{1}{2}(p_1 - p_2) = \sqrt{(\text{E N}^2 + \text{N C}^2)} = \sqrt{(\tfrac{1}{4}(p_x - p_y)^2 + q^2)}.$$

Add and subtract.

$$p_1 = \text{O E} + \text{E C} = \tfrac{1}{2}(p_x + p_y) + \sqrt{(\tfrac{1}{4}(p_x - p_y)^2 + q^2)},$$

$$p_2 = \text{O E} - \text{E C} = \tfrac{1}{2}(p_x + p_y) - \sqrt{(\tfrac{1}{4}(p_x - p_y)^2 + q^2)}.$$

The new major principal unit stress p_1 will bisect the angle N E C, and the new minor principal stress p_2 will be at right angles to p_1. Therefore

$$\text{Tan } 2\,\theta = \tan \text{N E C} = q \div \tfrac{1}{2}(p_x - p_y),$$

from which θ, the obliquity to O N, can be found. The two new principal stresses with their planes are represented in the figure.

If E C is less than O E, p_2 will have the same sign as p_1; if greater, p_2 will be opposite in sign to p_1.

Example.—If $p_x = 5{,}000$ lbs., $p_y = 1{,}000$ lbs., $q = 3{,}000$ lbs., then $p_1 = 7{,}200$, $p_2 = -600$, $\theta = 28°\ 09'$. If $p_x = -5{,}000$ lbs., the values change to 2,440, — 6,440 and 22° 30′.

Bending and torsion combined give rise to a construction for maximum stress like Fig. 76, p_y being usually taken equal to zero. There results $p_1 = \frac{1}{2}\,p_x + \sqrt{(\frac{1}{4}\,p_x^{\,2} + q^2)}$, from which value of max. stress is derived, in § 93, the max. necessary resisting moment or the required section of a beam or shaft subjected to bending and torsion together.

198. Local Loading.—The above investigation also applies to the problem of local loading on a beam. In the preceding Fig. 76, p_y may denote the unit pressure on the horizontal plane at a certain point in a beam due to a heavy local load, if the load is on top, or a tension if the load is below the beam. If the load is applied elsewhere in the vertical, there will be pressure on horizontal planes below and tension on those above the load.

As to the magnitude of p_y :—A load W at any point of the beam goes to the two points of supports as shear. This shear is distributed over the cross-section of a beam according to the law developed in § 86. The pressure, or tension, on any horizontal plane F E, Fig. 35, p. 75, at a loaded point, will be that due to W less the amount of shear carried by the section of the fibres G E or E A on that side of this horizontal plane on which W is placed. It is doubtful whether the effect is of much importance so long as W does not injure the material.

199. Axes of Direct Stress in a Beam.—The stress at one edge of any right section of a beam will be normal and compressive; at a point a little nearer the neutral axis, the normal stress will be a little less and a small unit shear will also be found, with the result that the direction of the real principal stress at that point is slightly inclined to the plane of the section. The direct stress decreases regularly as the centre is approached, the unit shear increasing, § 86, and at the neutral axis there is only shear on two planes at right angles, one longitudinal, the other transverse. These shears are equivalent to normal tension and compression on planes at 45° with the axis. Hence the tangents at different points of curves such as represented in Fig. 77 will show the direction of the principal tension or compression at such points.

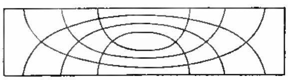


Fig. 77.

At the section of max. bending moment, at which section the shear will be zero, the curves will be horizontal. They will cross the neutral axis at 45° and will be vertical when they reach the opposite side of the beam. The stress diminishes along the curve, being maximum at the section of maxi-

mum bending moment, equal to the unit shear at the neutral axis, and zero at the edge of the beam.

200. Change of Volume.—If p = unit stress per square inch on the cross-section of a prism, and λ is the resulting stretch or shortening *per unit of length*, then by definition $E = p \div \lambda$, if p does not exceed the elastic limit.

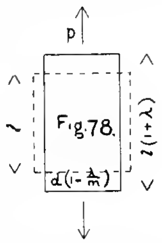
When a prism is extended or compressed by a simple longitudinal stress, it contracts or expands laterally, Fig. 78. This contraction or expansion per unit of breadth may be written $\mp \lambda \div m$, where $1 \div m$, the ratio of lateral contraction to longitudinal extension, is a constant for a given material, and for most solids lies between 2 and 4. § 205.

A simple longitudinal tension p then accompanies a

Longitudinal stretch $= \lambda = p \div E$ per unit of length, and a
Transverse contraction $= -\lambda \div m = -p \div mE$ per unit of breadth.

For ordinary solids λ is so small that it makes no difference whether it is measured per unit of original or per unit of stretched length. The original length will be used here.

The new length of the prism is $l(1 + \lambda)$ and the cross-section is $S(1 - \lambda \div m)^2$. The volume has changed from Sl to $Sl(1 + \lambda - 2\lambda \div m)$ nearly, if higher powers of λ than the first are dropped, since the unit deformations are very small. The change of unit volume is therefore $\lambda \left(1 - \dfrac{2}{m}\right)$. Thus, if m is nearly 4, for metals, the change of volume of one cubic unit is $\frac{1}{2}\lambda$ nearly, the volume being *increased* for longitudinal tension. If there were no change of volume, m would be 2, as is the case for india rubber, for small deformations. Similarly, for compression the change of unit volume is nearly $-\frac{1}{2}\lambda$ for metals, the volume being *diminished*.



Example.—Steel, $E = 29,000,000$; $p = 20,000$ lbs. per sq. in. tension; the extension will be $\dfrac{1}{1,450}$ of its initial length, the lateral contraction will be about $\dfrac{1}{5,800}$ of its initial width, and its increase of volume about $\dfrac{1}{2,900}$.

201. **Effect of two Principal Stresses.**—Denote the stresses by p_1 and p_2, treated as tensile. If they are compressive, reverse the signs.

Under the action of p_1 there will be the following stretch of the sides per unit of length:—Fig. 79.

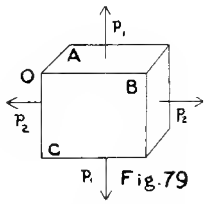


Fig. 79

$$\text{Parallel to O C, } \frac{p_1}{E};$$

$$\text{parallel to O B and O A, } -\frac{p_1}{mE}.$$

Under the action of p_2 there will be

$$\text{Parallel to O B, } \frac{p_2}{E};$$

$$\text{parallel to O A and O C, } -\frac{p_2}{mE}.$$

Adding the parallel changes or stretches

$$\text{Parallel to O C, } \lambda_1 = \frac{1}{E}\left(p_1 - \frac{p_2}{m}\right);$$

$$\text{Parallel to O B, } \lambda_2 = \frac{1}{E}\left(p_2 - \frac{p_1}{m}\right);$$

$$\text{Parallel to O A, } \lambda_3 = -\frac{1}{mE}(p_1 + p_2).$$

If p_1 and p_2 are equal unit stresses, but of opposite signs, the changes of length become

$$\frac{p}{E}\left(1 + \frac{1}{m}\right); \quad -\frac{p}{E}\left(1 + \frac{1}{m}\right); \quad \text{and zero;}$$

or, putting either of these two changes equal to λ, the lengths of the sides of the cube originally unity per edge will be $1 + \lambda$, $1 - \lambda$ and 1, and the volume, neglecting λ^2, is unchanged.

202. **Effect of two Shears.**—By § 189, two equal principal stresses of opposite signs are equivalent to two unit shears of the same amount per square inch, on planes at 45° with the original axes. Hence the above *distortion* results from two shears at right angles, and necessarily equal; and such shears cause *no change of volume.*

Fig. 80 shows the distorted cube. A square traced on the side of the original cube will become a rhombus, the angles

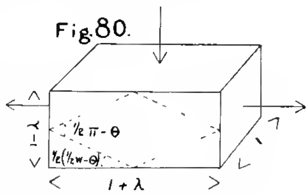
of which are greater and less than a right angle by the equal amount θ. Now one-half the angle $\frac{1}{2}\pi - \theta$ has for its tangent $\frac{1}{2}(1 - \lambda) \div \frac{1}{2}(1 + \lambda)$; hence

$$\frac{1 - \lambda}{1 + \lambda} = \tan \tfrac{1}{2}(\tfrac{1}{2}\pi - \theta) = \frac{1 - \tan \frac{1}{2}\theta}{1 + \tan \frac{1}{2}\theta}; \text{ or } \lambda = \tan \tfrac{1}{2}\theta.$$

But as θ is small, $\lambda = \frac{1}{2}\theta$, or $\theta = 2\lambda$.

Therefore a stretch and an equal shortening, along a pair of rectangular axes, are equivalent to a simple distortion relatively to a pair of axes making angles of 45° with the original axes; and the amount of the distortion is double that of either of the direct changes of length which compose it. This fact also appears from the consideration that a distortion of a square is equivalent to an elongation of one diagonal and a shortening of the other in equal proportions.

Fig. 80.



For steel, as before, $\lambda = \frac{1}{1,450}$, $\theta = \frac{1}{725} = 4' \, 45''$, if $p_1 = - p_2 = 20,000$ lbs.

203. Modulus of Shearing Elasticity.—Similarly, equal shearing stresses q on two pairs of faces of a cube, in directions parallel to the third face, will distort that third face into a rhombus, each angle being altered an amount θ, there being distortion of shape only, and not change of volume, Fig. 81.

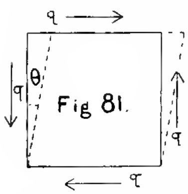


Fig. 81.

Under the law which has been proved true within the elastic limit, and the definition of the modulus of elasticity, § 10, a modulus of transverse (or shearing) elasticity, C, also called co-efficient of rigidity, as E may be called co-efficient of stiffness, may be written, $C = q \div \theta$.

As these two unit shears are equivalent to a unit pull and thrust of the same magnitude per square inch, at right angles

with each other and at 45° with these shears, the case is identical with the preceding one. Then

$$\theta = 2\lambda, \text{ and } \lambda = \frac{p}{\mathrm{E}}(1 + \frac{1}{m}). \quad \therefore \theta = \frac{2p}{\mathrm{E}} \cdot \frac{m+1}{m}.$$

$$\text{But, as } p = q = \mathrm{C}\,\theta,\ \mathrm{C} = \frac{p}{\theta} = \tfrac{1}{2} \cdot \frac{m\mathrm{E}}{m+1}.$$

For iron and steel m is nearly 4, which gives $\mathrm{C} = \frac{2}{5}\mathrm{E}$. For wrought iron and steel, C is one or two one-hundredths less than 0.4E. Some use $\frac{3}{8}$E. $\mathrm{C} = 11,000,000$ is a fair value. § 205.

204. Stresses Resolvable into Shear and Fluid Pressure.—All systems of stress acting on a body may be resolved into distorting or shearing stresses, which do not alter the volume, and a stress p like fluid pressure, equal in all directions and normal, but positive or negative.

Suppose a cube of unit length of side acted on by a normal stress p_1 on two opposite faces. It will in no way alter the conditions of stress to apply $+p$ and $-p$ normally to each of the four remaining faces of the cube, and to make $p = \frac{1}{3}p_1$, as seen in Fig. 82. A $+p$ on each horizontal face and a $-p$ on two opposite vertical faces together form a pair of shearing or distorting stresses; another $+p$ on each horizontal face and a $-p$ on the other two vertical faces act similarly. These shears produce no change of volume. There remains, therefore, a stress of $+p = \frac{1}{3}p_1$, normal on every face, in all constituting a fluid stress, which will increase or diminish the volume, according to the direction of p_1. Other stresses on the other faces may be looked at in the same light. Since a body has three principal axes of stress, take the cube parallel to these axes.

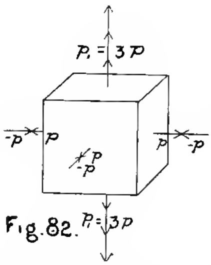


Fig. 82.

Example.—If $p_1 = 300$, $p_2 = -180$ and p_3, in the third coordinate direction $= -120$ lbs. per sq. in., $p = \frac{1}{3}(300 - 180 - 120) = 0$, and there will be no change of volume. If $p_1 = 450$ lbs. pressure, $p_2 = p_3 = 125$ lbs. pressure, the change of volume per cubic inch will be due to $233\frac{1}{3}$ lbs. per sq. in. on each face.

205. **Coefficient of Elasticity of Volume.**—Let v be the change of volume per cubic inch of the body under a normal unit stress p on any and all areas.

Then $K = p \div v$ is called the coefficient of elasticity of volume. The relation between K and the other constants can now be easily found.

A *simple* normal stress $3p$, by § 200, will produce a change of volume $3\lambda\left(1 - \dfrac{2}{m}\right)$. But, the shears being discarded, a simple normal stress $p_1 = 3p$ produces the same change of volume as a normal stress $p = \frac{1}{3}p_1$ on any and all areas of the cube. Hence

$$K = p \div 3\lambda\left(1 - \frac{2}{m}\right); \text{ or, since } \lambda = \frac{p}{E},$$

$$K = E\,\frac{m}{3m - 6}. \quad \text{But, by § 203, } E = 2C\,\frac{m+1}{m};$$

$$\therefore \quad K = 2C\,\frac{m+1}{3m-6}. \qquad m = \frac{6K + 2C}{3K - 2C}. \qquad E = \frac{9CK}{3K + C}.$$

	K	C	E	m
Brass	14,250,000 / 15,400,000	4,900,000 / 5,700,000	13,500,000 / 20,000,000	3.0
Copper	23,900,000	6,300,000 / 7,000,000	16,700,000 / 17,500,000	2.6
Cast Iron	13,700,000	7,500,000	19,000,000	3.7
Wrought Iron	20,700,000	11,000,000	28,000,000	3.6
Steel	26,200,000	11,600,000	29,000,000 / 34,800,000	3.25
Timber		75,000 / 100,000	800,000 / 1,400,000	

C may be found from the torsional vibrations of a wire. If t = time of a single oscillation in seconds, I = moment of inertia of the vibrating system about its axis of rotation, and T = twisting couple, $t = \pi\sqrt{(I\,\theta \div T)}$. Then §§ 89, 91,

$$T = \tfrac{1}{2}\pi q_1 r_1^3; \; q_1 = C r_1\,\frac{\theta}{l}, \; T = \tfrac{1}{2}\pi C r_1^4\,\frac{\theta}{l}, \text{ and } C = \frac{2\pi I l}{t^2 r_1^4}.$$

206. **Stress on One Plane the Cause of Other Stresses.**—The elongation produced by a pull, the shortening produced by a thrust, and the distortion due to a shear can be laid off as graphical quantities and discussed as were unit stresses themselves. All the deductions as to stresses have their counterparts in regard to changes of form. There has

been found an ellipse of stress for forces in one plane, when two stresses are given. Also, when three stresses not in one plane are given, there is an ellipsoid of stress which includes all possible unit stresses that can act on planes in different directions through any point in a body. So there is an ellipse or ellipsoid that governs change of form.

Whether the movement of one particle towards, from or by its neighbor sets up a resisting thrust, pull or shear, or the application of a pressure, tension or shear is considered to cause a corresponding compression, extension or distortion, the stresses and the elastic change of form coexist. Hence it follows that, when a bar is extended under a pull and is diminished in lateral dimensions, a compressive stress acting at right angles to the pull must be aroused between the particles, and measured per unit of area of longitudinal planes, together with shears on some inclined sections.

That such a state of things can exist may be seen from the following suggestions. It may be conceived that the particles of a body are not in absolute contact, but are in a state of equilibrium from mutual actions on one another. They resist with increasing stress all attempts to make them approach or recede from each other, and, if the elastic limit has not been exceeded, they return to their normal positions when the external forces cease to act. The particles in a body under no stress may then be conceived to be equidistant from each other. The smallest applied external force will probably cause change in their positions.

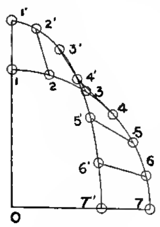


Fig 83.

If, in the bar to which tension is to be applied, a circle is drawn about any point, experiment and what has been stated about change of form in different directions show, that the diameter in the direction of the pull will be lengthened when the force is applied, the diameter at right angles will be shortened, and the circle will become an ellipse. In Fig. 83, particle 1 moves to 1′, 2 to 2′, 4 to 4′ and 7 to 7′. As they were all equidistant from 0 in the beginning, 1 in moving to 1′ offers a tensile resistance, 7 resists the tendency to approach

o, while a particle near 4, moving to 4′, does not change its distance from o, but moves laterally, setting up a shearing stress. A sphere will similarly become an ellipsoid.

207. **Actual Resulting Stresses.**—Let $-p_1$ be the unit tension in the direction o-1, and $+p_2$ the accompanying unit thrust in the direction 7-o. If a pull is applied to the solid in the direction o-7, which develops $-p'_2$ tension in that direction on the plane o-1, and p'_1 thrust in the direction 1-o, the resultant unit tension along o-1 on the plane o-7 will be $-p_1 + p'_1$, and along o-7 on the plane o-1 will be $-p'_2 + p_2$. It follows that the tension in the direction o-1 will be less than when the first pull was acting alone. Hence a plate is stronger to resist two pulls at right angles than when subjected to one only. The opposite deduction can be drawn if one principal stress is of opposite sign to the other.

A boiler plate has a tension in a tangential direction, that is, on a linear inch of longitudinal element, of pr, or of $pr \div t$ per square inch, where p = steam pressure per square inch, r = radius in inches, and t = thickness of plate. On a circumferential inch the pull is one-half as much. Then, by § 200, and what has been stated above,

$$-p_1 = pr, \quad p_2 = pr \div m = \tfrac{1}{4}p_1 = \tfrac{1}{4}pr,$$
$$-p'_2 = \tfrac{1}{2}pr, \quad p'_1 = \tfrac{1}{4}p'_2 = \tfrac{1}{8}pr.$$

Hence $pr - \frac{1}{8}\,pr = \frac{7}{8}\,pr$, or the true unit tension is less than the apparent tension by $12\frac{1}{2}$ per cent., and the boiler is stronger than it would be if the longitudinal tension from the steam pressure on the heads did not exist.

If tension is applied to the ends of a rectangular prism, and external compression is added to all four sides, the true unit tension is much increased, or the piece is decidedly weaker in resisting the pull.

Example.—At a certain point in a conical steel piston there exist principal stresses of 3,160 and 1,570 lbs., of opposite signs. Then $-p_1 = 3{,}160$, $p_2 = \frac{1}{4}p_1 = 790$. $p'_2 = 1{,}570$, $p'_1 = \frac{1}{4}p'_2 = 390$. $p_1 + p'_1 = 3{,}550$; $p_2 + p'_2 = 2{,}360$.

Since test experiments to determine tensile and compressive strength are made by the application of a single direct force, the values so determined are compatible with the

existence of the opposite stress on planes at right angles with the cross-section. Hence the working stresses for any material may fairly be considered to be a little higher than ordinary experiments show, provided account is taken at the same time of all the stresses which act on a particle.

208. **Cooper's Lines.**—Steel plate as it comes from the mill has a firmly adhering but very brittle film of oxide of iron on the surface. This film is dislodged by the extension of a test specimen in tension when the yield point is passed. If a hole is punched at moderate speed in a steel plate, so that the particles under the punch have some opportunity to flow laterally under the compression, there will be a radial compressive stress in all directions outwardly from the circumference of the hole. The unit compressive stress will rapidly diminish as the circumference is left behind, and points will soon be reached where tension at right angles will be set up. If there is lateral crowding at points near the circumference there may be lateral compression. Presently at a certain distance the tension will be one-fourth the compression. Then, from the ellipse of stress, if $p_2 = \frac{1}{4} p_1$ and is of contrary sign, $\frac{1}{2}(p_1 + p_2) = \frac{3}{8} p_1$; and $\frac{1}{2}(p_1 - p_2) = \frac{5}{8} p_1$, and shearing planes will exist, lying through the points where a circle of radius $\frac{1}{2} p_1$ cuts the ellipse. If p_2 is less than $\frac{1}{4} p_1$ the shearing planes will lie nearer the direction of p_1, and if $p_2 = p_1$, the shearing planes will make $45°$ with p_1. The scale breaks on these lines of shear and there result curves where the bright metal shows through, branching out from the hole, intersecting and fading away. The process of shearing a bar will develop the same curves from the flow of the metal on the face at the cut end. They are known as Cooper's lines.

These lines show that deformation takes place at considerable distances from the immediate point of shearing or punching.

Examples.—1. A pull of 1,000 lbs. per sq. in. and a thrust of 2,000 lbs. per sq. in. are principal stresses. Find the kind, direction and magnitude of the stress on a plane at $45°$ with either principal plane.

2. Find the stress per running unit of length of joint for a spiral riveted pipe when the line of rivets makes an angle of $45°$ with the axis of the pipe and when it makes an angle of $60°$.

$0.707\ p$; $0.5\ p$.

3. A rivet is under the action of a shearing stress of 8,000 lbs. per sq. in. and a tensile stress, due to the contraction of the rivet in cooling, of 6,000 lbs. per sq. in. Find p_1 and p_2.

$p_1 = -11,540$ lbs; $p_2 = +5,540$ lbs.

4. A connecting plate to which several members are attached, as IV., Plate III., has a unit tension on a certain section of 6,500 lbs. at an angle of 30° with the normal. On a plane at 60° with the first plane the unit stress of 5,000 lbs. compression is found at 45° with its plane. Find the principal unit stresses and the shear.

— 6,600; + 4,800; 5,700.

5. Assuming the weight of earth to be 105 lbs. per c. ft. and the horizontal pressure to be one-third the vertical, what is the direction and unit pressure per sq. ft. on a plane making an angle of 15° with the vertical at a point 12 ft. under ground, if the surface is level? 515 lbs.; $39\frac{1}{2}°$ with the horizon.

6. A stand-pipe, 25 ft. diam., 100 ft. high. The tension in lowest ring, if $\frac{7}{8}$ in. thick, is 7,440 lbs. per sq. in. If plates range regularly from $\frac{7}{8}$ in. thick at base to $\frac{1}{4}$ in. at top, neglecting lap, the compression at base will be about 215 lbs. per sq. in. For a wind pressure of 40 lbs. per sq. ft., reduced 50% for cylindrical surface, and treated as if acting on a vertical section, M at base = 2,500,000 ft. lbs. Compression on leeward side at base = 485 lbs. per sq. in. If $p_1 = -7,440$ lbs., $p_2 = 215 + 485 = 700$ lbs., find the stress and its inclination for a plane at 30° to the vertical? + 6,193 lbs.; 34° 40′.

Prove that the shearing plane is 17° 04′ from horizontal, and that the shear is 2,284 lbs. per sq. in.

CHAPTER XII.

RIVETS: PINS.

209. **Riveted Joints.**—There are four different ways in which riveted joints and connections may fail. The rivets may shear off; the hole may elongate and the plate cripple in the line of stress; the plate may tear along a series of rivet holes, more or less at right angles to the line of stress; or the metal may fracture between the rivet hole and the edge of the plate in the line of stress. From the consideration that a perfect joint is one offering equal resistance to each of these modes of failure, the proper proportions for the various riveted connections are deduced.

210. **Resistance to Shear.**—The safe resistance of a rivet to shearing off depends upon the safe unit shear and the area of the rivet cross-section, which varies as the square of the diameter of the rivet. When one plate is drawn out from between two others, a rivet is sheared at two cross-sections at once, and is twice as effective in resisting any such action. Rivets so circumstanced are said to be in double shear, and their number is determined on that basis.

211. **Bearing Resistance.**—The resistance against elongation of the hole or crippling the plate depends on the safe unit compression and what is known as the *bearing* area, the thickness of the plate multiplied by the semi-circumference of the hole. As the semi-circumference varies as the diameter, it is more convenient, and sufficiently accurate, to use the product of the thickness of the plate and the diameter of the rivet with a value of allowable unit compression about fifty per cent. greater than usual.

212. **Resistance of Plate.**—The resistance to tearing across the plate through a line of holes, or in a zigzag through two lines of holes in the same approximate direction, depends on the safe unit tensile stress multiplied by the cross-section

of the plate after deducting the holes. If the transverse *pitch*, or distance between centres of rivets, is considerable, an assumption of uniform distribution of tension on that cross-section is not likely to be true.

The resistance of the metal between the rivet hole and the edge of the plate in the line of stress is usually taken as the safe unit shear for the plate multiplied by the thickness and twice the distance from the rivet hole to the edge. Some, however, consider that the resisting moment of the strip of metal in front of the rivet holes is called into action.

213. **Bending: Friction.**—There are those who advise the computing of a rivet shank as if it were subjected to a bending moment. If the rivet fills the hole and is well driven, there is no bending moment exerted on it, unless it passes through several plates. As practical tests have shown that rivets cannot surely be made to fill the holes, if the combined thickness of the plates exceeds five diameters of the rivet, this limitation will diminish the importance of the question of bending.

No account is taken of the friction induced in the joint by its compression and the cooling of the rivet, and such friction gives added strength. As the rivet is closed up hot, the shank is under more or less tension when cold; the head is not given the thickness required in the head of a bolt under tension. Therefore rivets are not available for any more tension, and should not be used for that purpose. Tight-fitting, turned bolts are required in such a case.

214. **Spacing.**—The rivets should be well placed in a joint or connection, in order to insure a nearly uniform distribution of stress in the piece; they should be symmetrically arranged, be placed where they can be conveniently driven, and be spaced so that the holes can be definitely and easily located in laying out the work, See Plate III.

215. **Minimum Diameter of Rivets.**—The punch must have a little clearance in the die. The wad of metal shears out below the punch with more ease and with less effect on the surrounding metal when it can flow, as it were, a little laterally, and it then comes out as a smooth frustum of a cone with hollowed sides, reminding one of the *vena contracta*.

See Plate I. The punch must also be a little larger than the rivet, to permit the ready entrance of the rivet shank at a high heat. The diameter of the hole is often computed at $\frac{1}{8}$ inch in excess of the nominal diameter of the rivet; but the rivet is treated as if of its nominal diameter.

One other consideration has weight in determining the minimum diameter of the rivet. If the rivet is of less diameter than the thickness of the plate, the punch will not be likely to endure the work of punching. A diameter one and a half times the thickness of the plate is often thought desirable.

216. Number and Size of Rivets.—Formulas are of little or no value in designing ordinary joints and connections. Boiler joints and similar work can be computed by formulas, but to no great advantage. Tables are used which give what is termed the shearing value of different rivet cross-sections in pounds, for a certain allowable unit shear, and the bearing or compression value of different thicknesses of plate and diameters of rivet, for a certain allowable unit compression. For a given thickness of plate, that diameter of rivet is the best whose two values, as above, most nearly agree. The quotient of the force to be transmitted through the connection or through a running foot of a boiler joint, divided by the less of the two practicable values will give the minimum number of rivets. Their distribution is governed by the considerations previously referred to. Whether a joint in a boiler requires one, two or three rows of rivets depends upon the number needed per foot.

Example.—Two tension bars, 6 in. by $\frac{1}{2}$ in., carrying 30,000 lbs., are to connected by a short plate on each side. Let unit shear be 7,500 lbs. per sq. in., unit compression 15,000 lbs., when diameter of rivet is used, and unit tension 12,000 lbs. The bearing value of $\frac{11}{16}$ in. rivet in a $\frac{1}{2}$ in. plate is 5,160 lbs., its shearing value in double shear is $2 \times 2{,}780 = 5{,}560$ lbs. A $\frac{5}{8}$ rivet would give 4,700 and 4,600 respectively, but is of rather small diameter for thickness of plate. Hence $30{,}000 \div 5{,}160 = 6$ rivets necessary. If these rivets can be so arranged that a deduction of but one rivet hole is necessary from the cross-section of the tie, $(6 - \frac{13}{16})\frac{1}{2} = 2\,\frac{19}{32}$ sq. in. net section, which will carry 31,125 lbs. at 12,000 lbs. unit tension. Each cover plate cannot be less than $\frac{1}{4}$ in. thick, and, as will be seen presently, should be made a little

more. The length will depend on the distribution of the rivets, to be taken up next.

217. **Arrangement of Rivets.**—Long joints, under tension, like those of boilers, are connected by one or more rows of rivets, as shown at A and B, Fig. 84. If more than one row is needed, the rivets are *staggered*, and the rows should be separated such a distance that fracture by tension is no more likely on a zigzag line than across a row. If the distance from centre to centre on the diagonal is not less than three-

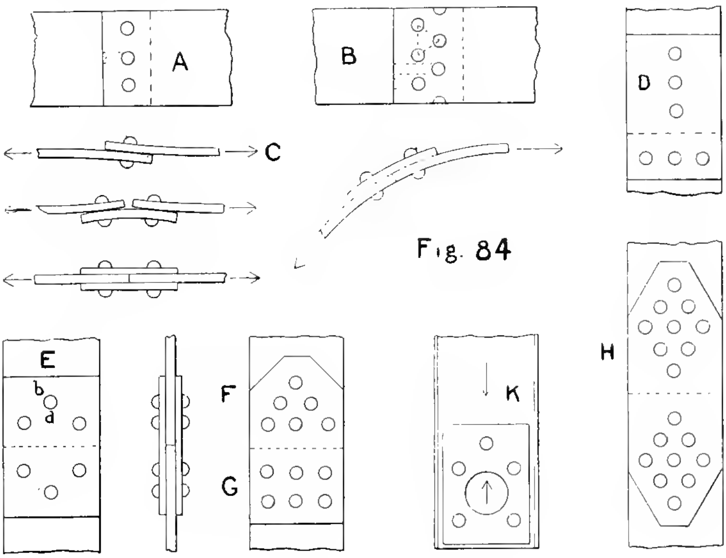


Fig. 84

fourths of the pitch, this object will be attained. To prevent tearing out at the edge of the plate, the usual specification of at least one and a half rivet diameters from centre of hole to edge of plate will suffice.

The *tendency* of a lap joint to cause an uneven distribution of stress by reason of bending, and the same tendency when a single cover is used, is shown at C. The increase of stress thus caused should be offset by increased thickness of plate. An outside cover strip on the joint of a cylinder exposed to internal pressure is not so bad, as the internal pressure tends to force the ring to conform to a circle. A

cover strip on either side is preferable, if not objectionable for other reasons.

In splicing ties, D shows a bad arrangement, the upper plan failing to distribute the stress evenly across the tie, and the lower plan wasting the section by excessive cutting away. The rivets at E are well distributed across the breadth, and weaken the tie by but one hole, as only two-thirds of the stress passes the section reduced by two holes; and, unless the net section at this place is less than two-thirds of the section reduced by one hole, it is equally strong. Thus $b - d = \frac{3}{2}(b - 2d)$, or a breadth equal to or greater than four diameters will satisfy this requirement. The covers, however, will be weakened by two holes, and hence their combined thickness, when two are used, should exceed the thickness of the tie.

F similarly is better than G, and the tie at F is again weakened by but one hole. To prevent the great weakening of the cover in this case the rivets may be spaced as shown at H; but it is doubtful whether the saving in thickness of the covers is not offset by the increase in length.

As it is desirable to transmit all but the proper fraction of the tension past the first rivet, the corners of the cover F or H are clipped off, thus increasing the unit tension in the reduced section and increasing its stretch to more nearly correspond with the unit tension and elongation of the tie beneath. The appearance of the connection is also improved.

218. **Remarks.**—If the member is in compression, the holes are not deducted, since the rivets completely fill the holes; and the strength is computed on the gross section. Unless special care is exercised in bringing two connected compression pieces into close contact at their ends, good practice requires the use of a sufficient number of rivets at the connection to transmit the given force.

Rivet heads in boiler work are flat cones. In bridge and structural work they are segments of spheres, known as button heads, and are finished neatly by means of a die. These heads may be flattened when room is wanting, and countersunk heads are used where it is necessary to have a finished flat surface.

Members which meet at an angle, are connected by plates and rivets. See Plate III. The axes of the several members should if possible intersect in a common point. If they do not, moments are introduced which give rise to what are known as secondary stresses, as distinguished from the primary stresses due to the direct forces in the pieces of the frame. Such secondary stresses may be of considerable magnitude in an ill-designed joint.

It is desirable to arrange the rivets in rows which can be easily laid out in the shop, and a central rivet, where several axes of pieces intersect, furnishes a convenient point of reference.

Commercial rivet diameters vary by sixteenths of an inch, more commonly by eighths, $\frac{5}{8}$, $\frac{3}{4}$, $\frac{7}{8}$ and one inch being the ones frequently used. As much uniformity as possible in the size of rivets will tend to economy in cost.

219. **Structural Riveting.** — The following rules for structural work are in harmony with good practice:—

Holes in steel $\frac{3}{8}$ inch thick or less may be punched; when steel of greater thickness is used, the holes shall be drilled.

Rivets shall have round concentric heads, of a depth at the circumference of the shank of not less than one-half the diameter of the rivet, and with full bearing on the plate.

The pitch of rivets, in the direction of the stress, shall never exceed 6 inches, nor 16 times the thickness of the thinnest plate connected, and not more than 30 times that thickness at right angles to the stress.

At the ends of compression members the pitch shall not exceed 4 diameters of the rivet, for a length equal to twice the width of the member.

The distance from the edge of any piece to the centre of a rivet hole must not be less than $1\frac{1}{2}$ times the diameter of the rivet, nor exceed 8 times the thickness of the plate; and the distance between centres of rivet holes shall not be less than 3 diameters of the rivet.

The diameter of the die shall not exceed that of the punch by more than $\frac{1}{16}$ of an inch, and all rivet holes shall be so accurately spaced and punched that, when the several parts are assembled together, a rivet $\frac{1}{16}$ inch less in diameter than the hole can generally be entered hot into any hole.

The *effective diameter* of a driven rivet will be assumed to be the same as its *diameter before driving;* but the *rivet hole* will be assumed to be *one-eighth inch* diameter *greater* than the undriven rivet.

220. **Boiler Riveting.**—The following proportions are sometimes used for tanks, stand-pipes, and similar work:—

Diameter of rivet, double the thickness of the plate. Pitch, centre to centre, 3 diameters of the rivet for a single row; 4 diameters for double or triple rows, with rivets staggered (zigzag) 3 diameters on the diagonal line. From centre of rivet line to the edge of plate, after it has been beveled to 60° for calking, $1\frac{1}{2}$ diameters $+ \frac{1}{8}$ inch.

Unwin gives the following rules for riveted joints:—

Single riveted lap joints:—Diameter of rivet, twice the thickness of plate; pitch of rivets and width of lap, three times the diameter of rivet. Butt joints with single cover:—The same as above.

Double riveted lap joints:—Diameter of rivet, twice the thickness of plate; pitch of rivets, four and a half times the diameter of the rivet; width of lap six diameters in zigzag riveting.

Butt joints with double covers, each cover being one-half the thickness of plate:—Diameter of rivet, one and a half times thickness of plate; pitch in single riveted joints, $3\frac{1}{4}$ diameters, and width of cover strips, 6 diameters; pitch in double riveted joints, 5 diameters, and width of cover strips, 12 diameters in zigzag riveting.

In ordinary cases there is no danger that the rivets will be too far apart to render the joint water or steam tight, when the edge of the plate on one or both sides is properly closed down with a calking tool.

221. **Strength of Joint.**—Rivet steel may be required to have an ultimate strength of from 54,000 to 62,000 lbs., a yield point of 30,000 lbs., and an elongation of 26 per cent. Similarly, rivet iron may show 50,000, 26,000, and 18 per cent.

The strength of a well-designed, single riveted joint may be 50 per cent.; of a double riveted joint, 65 per cent.; and of a triple riveted joint, 80 per cent. of that of the unpunched plate.

For unit stresses for shear and bearing see § 177.

222. Pins: Reinforcing Plates.—The pieces of a frame are frequently connected by pins instead of rivets. The axes of the several pieces are thus made to meet in a common point, if the pin hole is central in each member. Pins are subjected to compression on their cylindrical surfaces, to shear on the cross-section, and to bending moments. The compression on the pin-hole is reduced to the proper unit stress, if necessary, by riveting reinforcing plates to the sides of the members, as shown at K, Fig. 84. A sufficient number of rivets to transmit the proper proportion of the force must be used, with a due consideration of the shearing value of a rivet and its bearing value in the reinforcing plate or the member itself, which ever gives the less value. No more rivets should be considered as efficient behind the pin than the section of the reinforcing plate each side of the pin hole will be equivalent to.

When the pin passes through the web of a large built member such as a post or a top chord of a bridge, the web is often so thin that more than one reinforcing plate on either side is needed. It is then economical to make the several plates of increasing length, the shortest on the outside, and determine the number of rivets in each portion accordingly. The longest plate in such a case is sometimes required to extend in front of the pin four times the transverse distance from the pin centre to the line of rivets in the angle iron, so that the stress may be transferred to the flange angles and plate, and not overtax the web.

223. Shear and Bearing.—The shear at any section of the pin is found from the given forces in the pieces connected. The resultant of the forces in the pieces on one side of any pin section will be the shear at that section. As the pin will probably not fit the hole tightly (a difference of diameter of one-fiftieth of an inch being usually permitted), the maximum unit shear will be four-thirds of the mean, § 86. Specifications frequently give a reduced value for mean unit shear, which provides for this unequal distribution.

Bearing area is also figured as if projected on the diameter with, *e. g.*, 15,000 lbs. in place of 10,000 lbs. per sq. inch on the semi-circumference.

224. **Bending Moments on Pins.**—At a joint where several pieces are assembled, the resisting moment, required to balance the maximum bending moment on the pin caused by the forces in those pieces, will generally determine the diameter of the pin. In computing the bending moments, the centre line of each piece or bearing is considered the point of application of the force which it carries. This assumption is likely to give a result somewhat in excess of the truth, as any yielding tends to diminish the arm of each force.

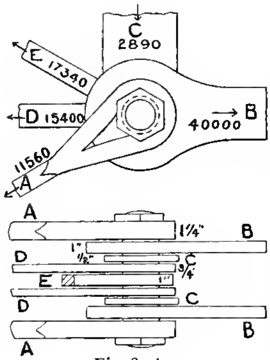


Fig. 84 A.

The process of finding the bending moments will be made clear by an illustration. Fig. 84, A, shows the plan and elevation of the pieces on a pin, with the forces and directions marked. The thickness of the pieces, which are supposed to be in contact, is also shown. The joint must be symmetrically arranged, to avoid torsion, and *simultaneous* forces must be used, which reduce to zero for equilibrium. As the joint is symmetrical, the computation is carried no farther than the piece adjoining the middle.

Resolve the given forces on two convenient rectangular axes, here horizontal and vertical. Set the horizontal components in order in the column marked H, the vertical ones in the column marked V. Their addition in succession gives the shears, marked F. The next column shows the distance from centre to centre of each piece. Fdx is then the increment of bending moment; and the summation of increments gives, in the column M, the bending moment at the middle of each piece, from the horizontal and from the vertical components respectively. The square root of the sum of the squares of any pair of component bending moments will be the resultant bending moment at that section. It is comparatively easy to pick out the pair of components which will give a maximum bending moment on the pin. Equate this value

with the resisting moment of a circular section and find the necessary diameter.

	H.	F.	*dx.*	F*dx.*	M.
A	+ 10,000	+ 10,000			
			$1\frac{1}{8}$		
B	— 40,000	— 30,000		+ 11,250	+ 11,250
			$\frac{3}{4}$		
C	0	— 30,000		— 22,500	— 11,250
			$\frac{5}{8}$		
D	+ 15,000	— 15,000		— 18,750	— 30,000
			$\frac{7}{8}$		
E	+ 15,000	0		— 13,125	— 43,125
	0				

	V.				
A	— 5,780	— 5,780			
			$1\frac{1}{8}$		
B	0	— 5,780		— 6,503	— 6,503
			$\frac{3}{4}$		
C	— 2,890	— 8,670		— 4,335	— 10,838
			$\frac{5}{8}$		
D	0	— 8,670		— 5,419	— 16,257
			$\frac{7}{8}$		
E	+ 8,670	0		— 7,586	— 23,843
	0				

M at D $= \sqrt{(30{,}000^2 + 16{,}257^2)}$; M at E $= \sqrt{(43{,}125^2 + 23{,}843^2)}$. The latter is plainly the larger, and is 49,210 in. lbs.

The pieces can be rearranged on this pin to give a smaller moment. The maximum moment is not always found at the middle.

The bending moment at any point of the beam or shaft, when the forces do not lie in one plane, can be found in the same way.

A solution of the above problem by graphics can be found in the author's Graphics, Part II., Bridge Trusses.

For values of unit stress in pins, see § 177.

For bolts, see § 134.

Examples.—1. A tie bar, ½ in. thick, and carrying 24,000 lbs., is spliced with a butt joint and two covers. If unit shear is 7,500 lbs., unit bearing on diam. is 15,000 lbs., and unit tension is 10,000 lbs., find the number, pitch and arrangement of ¾ in. rivets needed, and the width of the bar.

2. The longitudinal lap joint of a boiler must resist 52,000 lbs. tension per linear ft. If the unit working stress for the shell is 12,000 lbs. and the other stresses as above, what size of rivet is best, for double riveting, what the pitch, and the thickness of the shell?

3. A pin at VIII., Plate III., is to be computed. The force in the top chord at the right is 120,000 lbs., at the left 80,000 lbs., in the post 40,000 lbs., and the horizontal component of the tension in main tie, which slopes at 45°, is 40,000 lbs. The vertical plates of the chord are 12 in. apart, the two ties $9\frac{1}{2}$ in., and the side plates of the post, 8 in. If f for bending is 20,000, what is the diameter of the pin? Will the plates need reinforcing, if $\frac{1}{4}$ in thick?

CHAPTER XIII.

ENVELOPES.

225. **Stress in a Thin Cylinder.**—Boilers, tanks and pipes under uniform internal normal pressure of p per. sq. inch.

Conceive a thin cylinder, of radius r, to be cut by any diametral plane, such as the one represented in Fig. 85, and consider the equilibrium of the half cylinder, which is illustrated on the left. It is evident that, for unity of distance along the cylinder, the total pressure on the diameter, $2\,pr$, must balance the sum of the components of the pressure on the semi-circumference in a direction perpendicular to the diameter. This pressure $2\,pr$, uniformly distributed over the diameter, must cause a tension T in the material at each end to hold the diameter in place. Hence

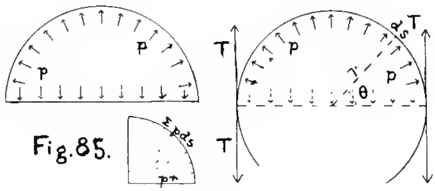


Fig. 85.

$$T = pr.$$

As all points of the circle are similarly situated, the tension in the ring at all points is constant, and equal to pr. If the thickness is multiplied by the safe working tension f per square inch, it may be equated with pr, giving,

$$\text{Required } net \text{ thickness} = pr \div f.$$

In a boiler or similar cylinder made up of plates an increase of thickness will be required to compensate for the rivet holes. If a is the pitch, or distance from centre to centre, of consecutive rivets in one row along a joint, and d' the diameter of the *rivet hole*, the effective length a to carry the tension is reduced

to $a - d'$, and the *gross* thickness of plate must not be less than $\frac{pr}{f} \cdot \frac{a}{a - d'}$.

Example.—The circumferential tension in a boiler, 4 ft. diam., carrying 120 lbs. steam pressure is $120 \cdot 24 = 2{,}880$ lbs. per linear inch of length of shell, which will require a plate $\frac{2{,}880}{10{,}000}$ in. thick (net), if f is not to exceed 10,000 lbs. per sq. in. Net thickness $= \frac{9}{32}$ in. If a longitudinal joint has $\frac{3}{4}$ in. rivet holes, at $2\frac{1}{4}$ in. pitch, in two rows, the thickness of plate must not be less than $\frac{2{,}880 \cdot 2\frac{1}{4}}{10{,}000 \cdot 1\frac{1}{2}} = \frac{7}{16}$ in.

226. Another Proof of the value of T may be obtained as follows:—The small force on arc $ds = pds$. The vertical component of this force $= pds \sin\theta = pdx$. The entire component on one side of the diameter is $\int_{-r}^{+r} pdx = 2\,pr$, which must be resisted by the tension in the material at the two ends of the diameter.

The same result will be obtained graphically by laying off a load line $= \Sigma\, pds$, which becomes a regular polygon of an infinite number of sides, *i. e.*, a circle, with the lines to the pole making the radii of the length pr.

The cylinder, under these circumstances, is in stable equilibrium. If not perfectly circular, it tends to become so, small bending moments arising where deviation from the circle exists. Hence a lap joint in the boiler shell causes a stress from the resisting moment to be combined with the tension at the joint.

The above investigation applies only to cylinders so thin that the tension may be considered as distributed uniformly over the section of the plate.

For riveting see Chapter XII.

227. Stress in a Right Section.—The total pressure from p on a right section of the cylinder is $\pi r^2\, p$, which will also be the resultant pressure on the head in the direction of the axis of the cylinder, whether the head is flat or not. This pressure causes tension in every longitudinal element of the cylinder, or in every cross-section. As this cross-section is $2\,\pi r \times$ thickness, the longitudinal tension per linear inch of a circumferential joint is $\pi r^2\, p \div 2\,\pi r = \frac{1}{2}\, pr$, or one-half the

amount per linear inch of a longitudinal joint. Hence a boiler is twice as strong against rupture circumferentially as longitudinally. Hence, also, the longitudinal seams are often double riveted, while the circumferential ones are single riveted.

228. Stress in any Curved Ring under Normal Pressure.—Since the stress in a circular ring of radius r, under internal or external normal unit pressure p, is pr per linear unit of cross-section of the ring, and per sq. inch is $pr \div$ thickness, being tension in the first case and compression in the second case, the direct stress on the cross-section of any curved ring, at a point where the applied pressure is normal, will be given by the same expression, if for r is substituted the radius of curvature of the ring at that point. Unless the ring, however, has the form of equilibrium under the given applied forces, a resisting moment also is required at the cross-section; but the resultant force there may be found as stated.

Example.—An elliptic ring, diameters 60 in. and 30 in., has a normal pressure exerted on it, at the extremity of the shorter diameter, of 150 lbs. per linear inch of ring. The radius of curvature at that point is $a^2 \div b$, or $30^2 \div 15$, and the resultant force at that section will be $150 \cdot 30^2 \div 15 = 9{,}000$ lbs.

229. Thin Spherical Shell: Segmental Head.—If a thin hollow sphere of radius r' has a uniform normal unit pressure p applied to it within, the total interior pressure on a meridian plane will be $\pi r'^2 p$, and the tension per *linear* inch of shell will be

$$\pi r'^2 p \div 2\pi r' = \tfrac{1}{2} pr'.$$

If p is applied externally, the stress in the material will be compression. It may be noted that the *double* curvature of the sphere is associated with *half* the stress which is found in the cylinder of single curvature having the same radius.

If a segment of a sphere is used to close or cap the end of a cylinder or boiler, the same value will hold good. In this case the radius r' is greater than r for the cylinder.

If the segmental end is fastened to the cylinder by a bolted flange, the combined tension on the bolts will be $\pi r^2 p$, as this is the total force on a right section of the cylinder.

The flange itself will be in compression. The pressure p from below, in Fig. 86, causes a pull per circumferential unit, in the direction of a tangent at B, which pull has just

been shown to be equal to $\frac{1}{2}pr'$: It may be resolved into vertical and horizontal components. The vertical component B C is, by § 227, $\frac{1}{2}pr$. The horizontal component h must be proportioned to the vertical component as A O to A B, the sides of the right angled triangle to which they are respectively perpendicular. As A O $= \sqrt{(r'^2 - r^2)}$,

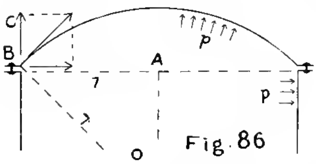


Fig. 86

$$h : \tfrac{1}{2}pr = \sqrt{(r'^2 - r^2)} : r,$$
$$\text{or } h = \tfrac{1}{2}p\sqrt{(r'^2 - r^2)}.$$

As h is a uniform normal pressure applied from without (or tension applied from within) in the plane of the flange, the compression on the cross-section of the latter will be hr or $\frac{1}{2}pr\sqrt{(r'^2 - r^2)}$, to be divided by that cross-section for finding the unit compression.

Segmental bottoms of cylinders are sometimes turned inward. The principles are the same.

Example.—A segmental spherical top to a cylinder of 24 in. diam., under 100 lbs. steam pressure, has a radius of 15 in. with a versed sine of 6 in. The tension in top $= \frac{1}{2}$ 100 . 15 $=$ 750 lbs. per linear inch. If its thickness is $\frac{1}{4}$ inch, the stress per sq. in. is 3,000 lbs. The total pull on the flange bolts is 100 . 144 . 22 $\div$ 7 $=$ 45,260 lbs. A $\frac{3}{4}$ in. bolt has about 0.3 sq. in. section at bottom of thread, giving a tension value of about 3,000 lbs. if $f =$ 10,000 lbs. There would be needed some 15 bolts, about $5\frac{1}{2}$ in. centre to centre on a circumference of 26 in. diameter. The compression in the flange is $\frac{1}{2}$. 100 . 12 . 9 $=$ 5,400 lbs. A 2 in. by $\frac{1}{2}$ in. flange, with a $\frac{3}{4}$ in. hole has a section $\frac{1}{2}$ $1\frac{1}{4}$ $= \frac{5}{8}$ sq. in., giving a unit compression in the flange of $\frac{8}{5}$. 5,400 $=$ 8,600 lbs. per sq. in.

A similar compression acts in the connecting circle between a water tank and the conical or spherical bottom sometimes built. See §§ 239, 240.

230. **Collapsing of Tubes.**—If a uniform normal pressure acts on a thin hollow cylinder from *without*, any ring is in unstable equilibrium, and any *slight deviation* from the circular form develops a bending moment, equal to pr multiplied by the deviation ordinate, which bending moment must be resisted by the ring. If the cylinder is quite thin, it has little ability to resist such a moment, and the cylinder or tube is in danger of collapsing. But, manifestly, if the cylinder is

closed at its ends, or is reinforced by rings or by internal diaphragms at intervals, the movement inwards at any point develops a resisting moment in the longitudinal elements. Hence a short, closed cylinder, or one with rings or flanges at proper intervals will be prevented from collapsing. The experimental relation between the pressure p which will cause collapsing, the length l, thickness t and diameter d of an iron tube, is given by Fairbairn as

$$p = 9,672,000\ t^2 \div ld \text{ nearly},$$

which appears to be, $p = Et^2 \div 3ld$, or $l = Et^2 \div 3pd$, all in inches.

For safety, l should be very much less.

The collapsing of an empty water tower or stand-pipe, under a strong wind pressure, is due to similar action, although, as the wind pressure is exerted on but one side, and is by no means uniformly distributed on the semicircle, the tendency to collapse is far greater. It is guarded against by one or more angle-iron rings riveted at or near the top. A 3 in. by 5 in. angle at the top will suffice for a tank 20 ft. in diameter; two similar angles for one 30 ft. in diameter; and bracing like a bicycle wheel will hold the top of larger tanks.

231. **Stand-Pipes.**—The following data for stand-pipes and water tanks are in keeping with good practice:—Use 57,000 to 65,000 lb. steel plates, showing 20 per cent. elongation in 8 inches. Allow $\frac{1}{8}$ inch for corrosion, and use no plates thinner than $\frac{1}{4}$ inch. The bottom of tanks may be $\frac{1}{4}$ or $\frac{5}{16}$ inch thick, unless greater strength is required from form of bottom or means of support; the bottom of stand-pipes may be $\frac{3}{8}$ or $\frac{1}{2}$ inch thick. Rivet heads should not be countersunk on the bottom, as full heads make tighter work. Bed the bottom on a fresh cement grouting on top of foundation Plates usually build 5 ft. rings. Alternate inside and outside rings are best. Internal angle-iron should be used at bottom, calked on both edges. If more bearing is wanted for the shell in large pipes, add outside angle. For riveting, see § 220.

See also, Engineering Record, Feb. 11, 1893.

Example.—Detailed calculation for riveting. A standpipe, 100 ft. high, 25 ft. diameter, full of water at $62\frac{1}{2}$ lbs. per c. ft. Let $f = 12{,}000$ lbs. tension and shear, and 10,000 lbs. compression, that is, 15,000 lbs. on diameter of hole, per sq. in. The tension per linear foot of the lowest ring will be $62\frac{1}{2} \cdot 100 \cdot 12\frac{1}{2} = 78{,}125$ lbs. $78{,}125$ lbs. $\div 12{,}000 = 6.51$ sq. in. of iron required per linear foot. As the final thickness will much exceed $\frac{1}{2}$ in., try $\frac{3}{4}$, $\frac{7}{8}$ and 1 in. rivets.

$\frac{3}{4}$ in., area =	0.442;	shearing value =	5,300 lbs.;	15 needed per ft.
$\frac{7}{8}$ " "	0.601	" "	7,220	11 " " "
1 " "	0.758	" "	9,430	$8\frac{1}{2}$ " " "

If 3 rows of $\frac{3}{4}$ in. rivets are used, 5 in a row, the pitch will be $2\frac{3}{8}$ in. The hole being called $\frac{7}{8}$ in. diameter, $5 \times \frac{7}{8} = 4\frac{3}{8}$ in. $12 - 4\frac{3}{8} = 7\frac{5}{8}$ in., net width of 1 ft. of sheet. $6.51 \div 7.62 = 0.854$ or $\frac{7}{8}$ in., required thickness of plate. $\frac{3}{4} \cdot \frac{7}{8} \cdot 15{,}000 = 9{,}844$ lbs., bearing value of 1 rivet. There are too many rivets for bearing, and the plate is too thick for $\frac{3}{4}$ in rivets.

If 3 rows of $\frac{7}{8}$ rivets are tried, $3\frac{2}{3}$ in a row per ft., the pitch will be $3\frac{1}{4}$ in. Hole is 1 in. diameter. $12 - 3\frac{2}{3} = 8\frac{1}{3}$ in. net width. $6.51 \div 8.33 = 0.78$ in. thickness of plate. A $\frac{3}{4}$ in. plate will serve, as pressure decreases upward on the joint, and the allowance for the hole is large. Eight rivets only are needed for bearing. If the distance between rows is made $2\frac{1}{2}$ in., the lap will be 8 in., or, for a cover strip, 16 in.

If 1 in. rivets are used, 3 rows with $4\frac{1}{4}$ in. pitch will be required, with $\frac{3}{4}$ in. plate, giving a bearing value per ft. of joint of 95,600 lbs.

So much detail is not necessary after a little experience.

232. **Thick Hollow Cylinder.**—If the walls of a hollow cylinder or sphere are comparatively thick, it will not be sufficiently accurate to assume that the stress in any section is uniformly distributed throughout it. If the material were perfectly rigid, the internal or external pressure would be resisted by the immediate *layer* against which the pressure was exerted, and the remainder of the material would be useless. As, however, the substance of which the wall is composed yields under the force applied, the pressure is transmitted from particle to particle, decreasing as it is transmitted, since each layer resists or neutralizes a portion of the normal pressure, and undergoes extension or compression in so doing.

233. **Greater Pressure on Inside.**—Let Fig. 87 represent the right section of a thick, hollow cylinder, such as that of an hydraulic press. Let r_1 and r_2 be the internal and

external radii in inches; p_1 and p_2 the internal and external normal unit pressures in pounds per square inch, p_1 being the greater; and p the unit normal pressure on any ring whose radius is r.

If a hoop is shrunk on to the cylinder, p_2 will be the unit normal pressure thus applied to the exterior of the cylinder.

The unit tensile stress found in a thin layer of radius r and thickness dr will be denoted by t, and will be due to *that portion of p which is resisted by the layer and not transmitted to the next exterior layer.* As the tension in a ring of radius r, under any interior normal unit pressure p is pr, the entire tension on a section from r_1 to r must be $p_1r_1 - pr$, which can also be expressed by $\int_{r_1}^{r} tdr$. As p and r are variables, there is obtained by differentiating the equation

$$p_1r_1 - pr = \int_{r_1}^{r} tdr,$$

$$-d\,(pr) = tdr,$$

$$\text{or } pdr + rdp + tdr = 0. \qquad (1.)$$

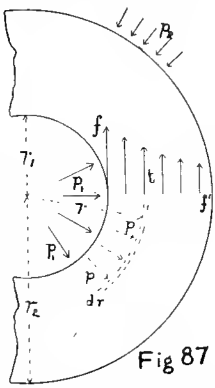


Fig 87

Another equation can be deduced from the enlargement of the cylinder. The fibres or layers between the limits r_1 and r, being compressed, will be diminished in thickness. The compression of a piece an inch in thickness by a unit stress p will be $p \div E$, § 10, and of one dr thick will be $pdr \div E$. The total diminution of thickness between r_1 and r, from what it was at first, will therefore be $\frac{1}{E}\int_{r_1}^{r} pdr$.

But the annular fibre or ring whose radius is r, and length $2\pi r$, has been elongated $t \div E$ per inch of length. Its length will now be $2\pi r\left(1+\frac{t}{E}\right)$ and its radius $r\left(1+\frac{t}{E}\right)$. The internal radius must similarly have become $r_1\left(1+\frac{f}{E}\right)$, where f is the value of t for radius r_1. The thickness $r - r_1$ has now become $r\left(1+\frac{t}{E}\right) - r_1\left(1+\frac{f}{E}\right)$, and, by sub-

tracting this value from $r - r_1$, there is found the diminution of thickness, $r_1 \frac{f}{E} - r \frac{t}{E}$. This expression may be equated with the previous one for decrease of thickness, or

$$r_1 \frac{f}{E} - r \frac{t}{E} = \frac{1}{E} \int_{r1}^{r} pdr.$$

Since the first term is constant, there is now obtained by differentiating this equation,

$$-d(tr) = pdr, \text{ or } tdr + rdt + pdr = 0. \tag{2.}$$

Add (1.) and (2.), and multiply by r to make a complete differential. Then integrate.

$$2(t + p)rdr + r^2(dt + dp) = 0$$

$$r^2(t + p) = \text{Constant}; \therefore = r_1^2(f + p_1) = r_2^2(f' + p_2). \tag{3.}$$

Again: subtract (1.) from (2.), and then integrate,

$$dt - dp = 0. \quad t - p = \text{Constant}; \cdot = f - p_1 = f' - p_2. \tag{4.}$$

From (3.) and (4.) are obtained, by addition and subtraction,

$$t = \frac{f - p_1}{2} + \frac{r_1^2}{r^2} \frac{f + p_1}{2}; \ p = -\frac{f - p_1}{2} + \frac{r_1^2}{r^2} \frac{f + p_1}{2}. \tag{5.}$$

If the internal radius is given, the external radius, and hence the required thickness, $r_2 - r_1$, is found by eliminating f' from (3.) and (4.)

$$r_2 = r_1 \sqrt{\left(\frac{f + p_1}{f - p_1 + 2p_2}\right)}. \tag{6.}$$

If p_2 is atmospheric pressure, it may be neglected when p_1 is large. In that case

$$r_2 = r_1 \sqrt{\left(\frac{f + p_1}{f - p_1}\right)}; \text{ or } r_2 - r_1 = r_1 \left(\sqrt{\frac{f + p_1}{f - p_1}} - 1\right).$$

As r_2 becomes infinite when the denominator of (6.) is zero, it appears that no thickness will suffice to bring f within the safe unit stress, if p_1 exceeds $f + 2p_2$.

These formulas do not apply to bursting pressures, nor to those which bring f above the elastic limit; for E will not then

be constant. They serve for designing or testing safe construction.

Examples.—Cylinders of the hydraulic jacks, for forcing forward the shield used in constructing the Port Huron tunnel, were of cast-steel, 12 in. outside diam., 8 in. diam. of piston, with $\frac{1}{4}$ in. clearance around same; pressure 2,000 lbs. per sq. in.

$$\frac{r_2^2}{r_1^2} = \frac{36 \cdot 16}{289} = \frac{f + 2{,}000}{f - 2{,}000}. \quad f = 6{,}030.$$

A cast-iron water-pipe, at the Comstock mine, was 6 in. bore, $2\frac{1}{2}$ in. thick, and was under a water pressure of 1,500 lbs. on the sq. in., or about 3,400 ft. of water. Here $f = 2{,}770$ lbs. per sq. in. for static pressure, while the formula for a thin cylinder gives 1,800 lbs.

234. Greater Pressure on Outside.—In this case the direction or sign of t will be reversed, it being compression in place of tension. From the preceding equations, without independent analysis, by making t negative, there result,

$$-\,d(pr) = -\,tdr; \quad d(tr) = pdr.$$
$$pdr + rdp - tdr = 0; \quad tdr + rdt - pdr = 0.$$
$$r^2(p - t) = r_1^2(p_1 - f) = r_2^2(p_2 - f').$$
$$t + p = f + p_1 = f' + p_2.$$

The outer radius and pressure will now be taken as given quantities, and the unit compression in the ring at any point will be

$$t = \frac{f' + p_2}{2} + \frac{r_2^2}{r^2} \cdot \frac{f' - p_2}{2}; \quad p = \frac{f' + p_2}{2} - \frac{r_2^2}{r^2} \quad \frac{f' - p_2}{2}. \quad (7.)$$

$$r_1 = r_2 \sqrt{\left(\frac{f - 2p_2 + p_1}{f - p_1}\right)}. \quad (8.)$$

which becomes, if p_1 is neglected as small,

$$r_1 = r_2 \sqrt{\left(1 - \frac{2p_2}{f}\right)}.$$

The external pressure p_2 must be less than $\frac{1}{2}(f + p_1)$, if r_1 is to have any value. It will be seen from t in (7.) that the compression is greatest at the interior.

Example.—An iron cylinder, 3 ft. internal diameter, resists 1,150 lbs. per sq. in. external pressure. The required thickness, if $f = 9,000$ lbs., is given by

$$18 = r_2 \sqrt{\left(1 - \frac{2,300}{9,000}\right)} = 0.86 r_2.$$

$r_2 = 20.9$ in. Thickness $= 3$ in.

235. Action of Hoops.—To counteract in a greater or less degree the unequal distribution of the tension in thick, hollow cylinders for withstanding great internal pressures, hoops are shrunk on to the cylinders, sometimes one on another, so that, before the internal pressure is applied, the internal cylinder is in a state of circumferential compression, and the exterior hoop in a state of tension. If the internal pressure on the hoop is computed, for a given value of f in the hoop, and this pressure is then used for p_2 on the cylinder, the allowable internal pressure p_1 on the cylinder consistent with a permissible f in this cylinder can be found. There is, however, an uncertainty as to the pressure p_2 exerted by the hoop.

Examples.—A hoop one inch thick is shrunk on a cylinder of 6 in. external radius and 3 in. internal radius, so that the max. unit tension in the hoop is 10,000 lbs. per sq. in. This stress, by § 233, will be due to an internal pressure on the hoop of 1,530 lbs. per sq. in.

$$\text{For } 7 = 6 \sqrt{\left(\frac{10,000 + p_1}{10,000 - p_1}\right)} \text{ or } \frac{49}{36} = \frac{10,000 + p_1}{10,000 - p_1}.$$

This external pressure p_2 on the cylinder will cause a compressive unit stress in the interior circumference of the cylinder when empty, after the hoop is shrunk on, of 4,080 lbs., and will permit an internal pressure in the bore of 8,448 lbs. per sq. in., consistent with $f = 10,000$ lbs. For $\dfrac{36}{9} = \dfrac{10,000 + p_1}{10,000 - p_1 + 3,060}$. The cylinder alone, without the hoop, would allow a value of p_1 given by $\dfrac{36}{9} = \dfrac{10,000 + p_1}{10,000 - p_1}$, or $p_1 = 6,000$ lbs. If the cylinder had been 4 in. thick, the internal pressure might have been 6,900 lbs. The gain with the hoop, for the same quantity of material, is 1,548 lbs., or some 22 per cent.

Hydraulic cylinder for a canal lift at La Louvière, Belgium, 6 ft. 9 in. interior diam., 4 in. thick, of cast-iron, hooped with steel. Hoops 2 in. thick, and continuous. When tested, before hooping, one burst with an internal pressure of 2,175 lbs. per sq.

in., one at 2,280 lbs., and a third at 2,190 lbs. These results, if the formula is supposed to apply at rupture, give an average tensile strength of 23,400 lbs. per sq. in. The hoops were supposed to have such shrinkage that an internal pressure of 540 lbs. per sq. in. would give a tension on the cast-iron of 1,400 lbs., and on the steel of 10,600 lbs. per sq. in. The ram is 6 ft. $6\frac{3}{4}$ in. diam., and 3 in. thick, of cast-iron, an example of the greater pressure outside.

236. Thick Hollow Sphere.—Greater pressure on inside. Let Fig. 87 represent a meridian section of the sphere. Suppose f, t, etc., to be perpendicular to the plane of the paper. The entire normal pressure on the circle of radius r_1 will be $p_1\,\pi r_1^2$, and the tension on the ring between radii r_1 and r will be $\pi\,(p_1 r_1^2 - pr^2)$. Any ring of radius r and thickness dr will carry $2\,\pi rt\,dr$, and hence is derived the first equation

$$\pi\,(p_1 r_1^2 - pr^2) = 2\,\pi \int_{r_1}^{r} rt\,dr, \text{ or } -d\,(pr^2) = 2\,rt\,dr.$$

$$\therefore \quad r^2\,dp + 2\,pr\,dr + 2\,rt\,dr = 0.$$

The second equation will be the same as obtained for the cylinder,

$$-d\,(tr) = pdr, \text{ or } rdt + tdr + pdr = 0.$$

Strike out the common factor r from the first equation, multiply the second by 2, and subtract.

$$2\,rdt - rdp = 0, \quad \text{or } 2\,dt - dp = 0.$$

$$2\,t - p = \text{Constant}; \quad \therefore = 2\,f - p_1 = 2\,f' - p_2. \qquad (9.)$$

Again: add the first to the second and multiply by r^2

$$r^3\,(dp + dt) + 3\,r^2\,dr\,(p + t) = 0. \quad \therefore$$

$$r^3\,(p + t) = \text{Constant}; \quad \therefore = r_1^3\,(f + p_1) = r_2^3\,(f' + p_2). \qquad (10.)$$

From (9.) and (10.),

$$t = \frac{2\,f - p_1}{3} + \frac{r_1^3}{r^3}\cdot\frac{f + p_1}{3}; \quad p = -\frac{2\,f - p_1}{3} + 2\,\frac{r_1^3}{r^3}\cdot\frac{f + p_1}{3}.$$

$$r_2 = r_1\,\sqrt[3]{\left(\frac{2\,(f + p_1)}{2\,f - p_1 + 3\,p_2}\right)}. \qquad (12.)$$

These formulas are not applicable to bursting pressures for the reason given before. For a finite value of r_2, p_1 must be less than $2\,f + 3\,p_2$. If p_2 is atmospheric pressure, it may be neglected, and

$$r_2 = r_1\,\sqrt[3]{\left(\frac{2\,(f + p_1)}{2\,f - p_1}\right)}.$$

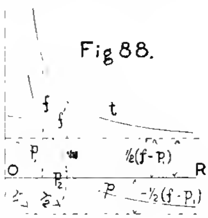
237. **Sphere: Greater Pressure on Outside.**—Here again t changes to compression or reverses in sign, yielding

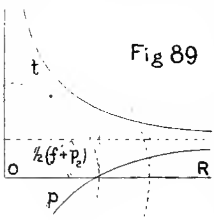
$$t = \frac{2f' + p_2}{3} + \frac{r_2^3}{r^3} \cdot \frac{f' - p_2}{3}\,; \quad p = \frac{2f' + p_2}{3} - 2\frac{r_2^3}{r^3} \cdot \frac{f' - p_2}{3}.$$

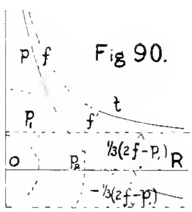
$$r_1 = r_2 \sqrt[3]{\left(\frac{2f + p_1 - 3p_2}{2(f - p_1)}\right)}. \tag{13.}$$

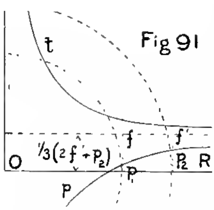
That r_1 shall be greater than zero requires that $p_2 < \frac{1}{3}(2f + p_1)$.

238. **Diagrams of Stress.**—Curves may be drawn to represent the variation of p and t in the four preceding cases. They are all hyperbolic, and, if r is laid off from the centre O on the horizontal axis, each curve will have the vertical axis through O for one asymptote, and for the other a line parallel to the horizontal axis, at a distance indicated by the first term in each value of t or p. The four accompanying sketches show the various curves. The values of f and f', the unit stresses in the material at the interior and exterior, which correspond to the given values of p_1 and p_2, are found at the extremities of the abscissas which represent r_1 and r_2. The error which would arise from considering t as uniformly distributed is manifest. The dotted circles show the respective cylinders or spheres. Fig. 88 gives the external and internal tensile stress for p_1 in the interior of a thick cylinder. Fig. 89 shows the distribution of compression when the greater pressure is from without. Figs. 90 and 91 represent thick spheres under similar pressures.









239. **Tank with Conical Bottom.**—A water tank of radius r may be built with a conical bottom and be supported at the perimeter only. Let the angle subtended by the cone

at the vertex be 2 θ, the distance along any element from the vertex D, Fig. 92, to any point P be a, and the depth of water above the vertex be h. Then the normal pressure at P, if w is the weight of a cubic unit of water, will be $w\,(h - a\cos\theta)$. If a plane of section is passed perpendicularly to the element D C at P, it will cut from the cone a conic section, usually an hyperbola. The radius of curvature, ρ, of that curve at its vertex P can be proved to be P I $= a \tan\theta$.*

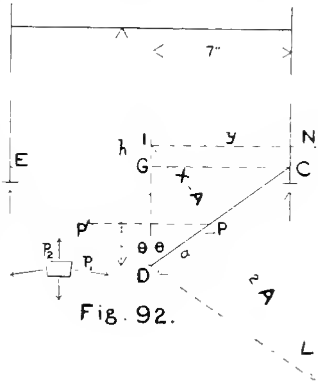


Fig. 92.

From the fact pointed out in § 228, that the stress, at a point in a curve where the external pressure is normal, is equal to the product of the unit pressure and the radius of curvature,—the *tension* at P, per unit of length of a *radial joint* along the element D C, will be

$$p\rho = w\,(h - a\cos\theta)\,a\tan\theta = wa\,(h\tan\theta - a\sin\theta).$$

As h is usually longer than D C, this tension will increase from D to C, being zero at D, and at C being equal to

$$w\,.\,\mathrm{D\,C}\,(h\tan\theta - \mathrm{D\,C}\sin\theta) = w\ \ \mathrm{D\,C}\,(h\tan\theta - r).$$

The tension in the radial joints will determine the thickness of the plates.

The load on the circumference, or any horizontal joint, cut out by a horizontal plane through P P′ will be made up of —the weight of the cylinder of water whose base is P P′, or $w\ \ \pi a^2\sin^2\theta\,(h - a\cos\theta)$; the weight of the cone P D P′ of water, or $w\ \ \pi a^2\sin^2\theta\,.\,\frac{1}{3}a\cos\theta$; and the weight of the metal cone below P P′, $w'\,.\,\pi a^2\sin\theta$, where $w' =$ weight of a square unit of plate. The last item is comparatively insignifi-

* By the calculus, ρ at vertex P $= \dfrac{B^2}{A} = \dfrac{A\,y^2}{x^2 - A^2}$. P L $= 2$ A; P I $= x -$ A; I N $= y$. $x - A = a\tan\theta$; $\tan(180^\circ - 2\theta) = \dfrac{2\,A}{a}$; $\therefore A = a\,\dfrac{\tan\theta}{\tan^2\theta - 1}$. $x + A = a\,\dfrac{\tan^3\theta + \tan\theta}{\tan^2\theta - 1}$. $y = (x - A)\sec\theta = a\,\dfrac{\tan\theta}{\cos\theta}$. $\therefore \rho = a\tan\theta$ = P I.

cant, but may be computed when the required thickness of plates has been found.

This weight

$$W' = w\ \pi a^2 \sin^2 \theta\ (h - \tfrac{2}{3} a \cos \theta)$$

must be carried by stress acting along the elements on the circumference P P′, that is, inclined along P C, at the angle θ. From a parallelogram of forces whose diagonal is W′, it may be seen that this total force on a circumference is W′ sec θ. The section over which it is distributed has a circumference $2\pi a \sin \theta$. The *tension* per unit of length of a *horizontal* section or *joint* will therefore be

$$\frac{w\ .\ \pi a^2 \sin^2 \theta\ (h - \tfrac{2}{3} a \cos \theta)}{\cos \theta\ .\ 2\pi a \sin \theta} = \tfrac{1}{2} wa \tan \theta (\ h - \tfrac{2}{3} a \cos \theta),$$

to which may be added $\tfrac{1}{2} w' a \sec \theta$ for the metal.

This value will be zero at D, and at C will be

$$\tfrac{1}{2} w\ .\ D\ C\ .\ \tan \theta\, (h - \tfrac{2}{3}\ D\ C\ .\ \cos \theta).$$

At C the tension is decomposed into a vertical component

$$v = \tfrac{1}{2} w\ .\ D\ C\ .\ \sin \theta (h - \tfrac{2}{3}\ D\ C\ .\ \cos \theta) = \tfrac{1}{2} wr (h - \tfrac{2}{3}\ D\ G)$$

per linear unit, carried by a circular girder which may itself be supported on a wall or on posts. The circular girder in the latter case acts as a beam with a torsional moment added.

The horizontal component at C will be

$$p' = \tfrac{1}{2} w\ .\ D\ C\ .\ \tan \theta \sin \theta (h - \tfrac{2}{3}\ D\ C\ .\ \cos \theta) = \tfrac{1}{2} wr \tan \theta (h - \tfrac{2}{3}\ D\ G),$$

which causes a compression of $p'r$ in the circular girder. As the tension in the tank ring at C is $w(h - D\ G)r$ per unit of length of a vertical joint, this tension serves to balance more or less of $p'r$, as the construction at C may permit.

The total weight on the circular girder is

$$w\pi r^2 (h - \tfrac{2}{3}\ D\ G) + w'\pi r\ .\ D\ C + \text{weight of tank}.$$

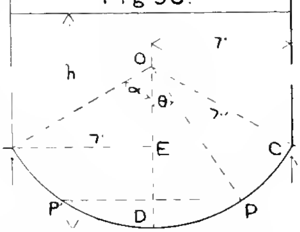
As the stresses found on the joints of the cone are principal stresses, there will be no greater ones at any point. The stress on any oblique plane can be readily found, if desired, by § 190.

Example.—A circular tank, 40 ft. diam. and 40 ft. high, has a conical bottom for which $\theta = 45°$ and $h = 60$ ft. Weight of c. ft. of water, 62.5 lbs.; length of element of cone 28.3 ft. Tension in radial joint at P, half way up, $= 62.5 \times 14.14(60 - 0.707 \times 14.14) = 44,187$ lbs. per linear ft. $= 3,682$ lbs. per in. of joint. Tension do. at C $= 62.5 \times 28.3(60 - 0.707 \times 28.3) = 70,700$ lbs. per ft. $= 5,892$ lbs. per in. of joint.

Tension in horizontal joint at P $= \frac{1}{2} \times 62.5 \times 14.14(60 - \frac{2}{3} \times 14.14 \times 0.707) = 23,567$ lbs. per ft. $= 1,964$ lbs. per in. of joint. Tension do. at C $= \frac{1}{2} \times 62.5 \times 28.3(60 - \frac{2}{3} \times 28.3 \times 0.707) = 41,242$ lbs. per ft. $= 3,437$ lbs. per in. of joint. $v = \frac{1}{2} \times 62.5 \times 20(60 - \frac{2}{3} \times 20) = 29,167$ lbs. per ft. of girder. As the horizontal component equals the vertical component, $p'r = 29,167 \times 20 = 583,340$ lbs. compression in the circular girder. Tension in lowest vertical ring of tank $= 62.5 \times 40 \times 20 = 50,000$ lbs. per linear ft.

240. Tank With Spherical Bottom.—If a cylindrical water tank, of radius r, has a segmental spherical bottom, of radius r', subtending a central angle of 2α, Fig. 93, the versed sine D E will be $r'(1 - \cos\alpha)$. $\sin\alpha = r \div r'$. Let the depth of water at the centre D be h.

Fig 93.



The *tension per unit of length* on any radial or *meridian joint* at any point P, the radius to which makes an angle θ with the vertical, will be, by § 229,

$$\tfrac{1}{2}w[h - r'(1 - \cos\theta)]r',$$

which become $\frac{1}{2}whr'$ at the bottom D, and at C is

$$\tfrac{1}{2}w[h - r'(1 - \cos\alpha)]\, r' = \tfrac{1}{2}w(h - \text{D E})r'.$$

A horizontal joint through P must support the weight of the cylinder of water of base P P′, or

$$w\pi r'^2 \sin^2\theta[h - r'(1 - \cos\theta)],$$

the weight of the water in the segment P D P′,

$$w\pi r'^2(1 - \cos\theta)^2[r' - \tfrac{1}{3}r'(1 - \cos\theta)]$$
$$= \tfrac{1}{3}w\pi r'^3(1 - \cos\theta)^2(2 + \cos\theta),$$

and the weight of metal in P D P′. If the weight of a square unit of plate is w', this last quantity is $2w'\pi r'^2(1 - \cos\theta)$,

and is comparatively insignificant. The weight of water in P D P′ may often be disregarded also.

The above vertical forces must be multiplied by cosec θ to give the force exerted, in the direction of a tangent, on the circumference P P′, and be divided by $2\pi r' \sin\theta$ to give the *tension per unit of length* of a *horizontal joint* at P, or

$$\tfrac{1}{2}wr'[h - r'(1 - \cos\theta)] + \tfrac{1}{6}wr'^2 \frac{(1 - \cos\theta)^2}{\sin^2\theta}(2 + \cos\theta)$$
$$= \tfrac{1}{2}wr'[h - r'(1 - \cos\theta) + \tfrac{1}{3}r' \tan^2 \tfrac{1}{2}\theta\,(2 + \cos\theta)].$$

At C substitute α for θ.

The vertical component, or load, at C, per unit of circumference on the circular girder will be found by dividing the weights given above by $2\pi r$ or $2\pi r' \sin\alpha$.

$$v = \tfrac{1}{2}wr[h - r'(1 - \cos\alpha)] + \tfrac{1}{6}wr'^2 \tan\tfrac{1}{2}\alpha(1 - \cos\alpha)(2 + \cos\alpha),$$

to which should be added the weight of the tank and of the bottom per unit of circumference, the latter being $w'r' \tan\frac{1}{2}\alpha$.

The horizontal component p', which causes a compression of $p'r$ in the circular girder, will be

$$p' = v \cot\alpha + w'r' \tan\tfrac{1}{2}\alpha \cot\alpha.$$

The stresses in the spherical bottom are smaller than those in the conical bottom.

Example.—A circular tank, 40 ft. diam. and 40 ft. high, has a spherical bottom for which $\alpha = 45°$. Then $h = 48.3$ ft., $r' = 28.3$ ft. Weight of a c. ft. of water, 62.5 lbs. Tension in radial joint at bottom $= \frac{1}{2} \times 62.5 \times 28.3 \times 48.3 = 42{,}645$ lbs. per linear ft. $= 3{,}554$ lbs. per in. of joint. Tension in radial joint at P, half way up, where $\theta = 22\frac{1}{2}°$, is 40,770 lbs. per ft., or 3,397 lbs. per in. of joint. At C, tension is 35,350 lbs. per ft., or 2,946 lbs. per in. of radial joint. Tension in horizontal joint at P is 41,760 lbs. per ft., or 3,480 lbs. per in., and at C is 39,220 lbs. per ft., or 3,268 lbs. per in. of joint. Compression in circular girder = 554,700 lbs.

241. Conical Piston.—If the cone E D C of Fig. 94 represents a conical piston of radius r, subtending an angle 2θ, with a uniform normal steam pressure p, per unit of area, applied over its exterior or interior, and if the supporting force is supplied by the piston rod at D, the compression or

tension exerted at P, on a radial section, per unit of length of D C, will be, by § 239, if D P $= a$, Fig. 92,

$$p_1 = p\,\rho = p \quad \text{I P} = pa \tan \theta;$$

and the unit stress will be found by dividing by the thickness t at P. The maximum unit stress is at C, and will be

$$\frac{pr}{t \sin \theta} \tan \theta = \frac{pr}{t} \sec \theta.$$

The vertical force on any horizontal or right section will be $p\pi\ (r^2 - a^2 \sin^2 \theta)$, which becomes at the vertex $\pi r'^2 p$, the force on the piston rod. This force will be compression on the rod and tension in the cone, if p acts on the exterior of the cone, and the reverse if p acts within the cone. The unit stress in the

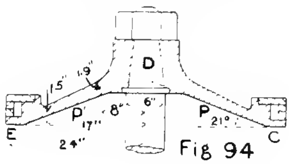


Fig 94

metal of the cone at this section will be found by multiplying this force by sec θ, and then dividing by the cross-section, $2\pi a \sin \theta \,.\, t$, giving

$$p_2 = \frac{p}{2at} \cdot \frac{r^2 - a^2 \sin^2 \theta}{\cos \theta \sin \theta} = \frac{p}{at} \cdot \frac{r^2 - a^2 \sin^2 \theta}{\sin 2\theta},$$

which is a maximum at the piston rod. If the radius of the rod is r', a then is $r' \div \sin \theta$, and the stress near the rod is

$$\frac{p}{2r't} \sec \theta\ (r^2 - r'^2).$$

As the stresses on a horizontal section and a radial section through any point will be of contrary signs at any given instant, there will be shearing planes at an angle with p_2, by § 192, whose tangent is $\sqrt{(p_2 \div p_1)}$, and the value of that shear will be $\sqrt{(p_1 p_2)}$.

Example.—Conical piston, Fig. 94, $r = 24$ in., $r' = 3$ in, $\theta = 69°$ Thickness, for $r' = 17$ in., is $1\frac{1}{2}$ in.; and, for $r' = 8$ in, is 1.9 in. Steam pressure, maximum difference on two sides, 100 lbs. per sq. in. For $r' = 17$ in., $p_1 = \dfrac{100 \,.\, 17 \,.\, 2}{3} \times 2.79 = 3162$ lbs. per sq. in.; $p_2 = \dfrac{100 \,.\, 2}{2 \,.\, 17 \quad 3} \times 2.79\ (24^2 - 17^2) = 1570$ lbs.; the shear $= \sqrt{(3162 \,.\, 1570)} = 2228$ lbs. per sq. in. on a plane making 35° 16′ with a radial element. For $r' = 8$ in., $p_1 = 1175$

lbs.; p_2 = 470 lbs.; and the shear = 743 lbs., at an angle of 32° 18′. For alternating stresses on steel castings, these values are satisfactory. See Example, § 207.

242. **Dome.**—A dome, subjected to vertical forces symmetrically placed around its axis, such as its own weight, may be treated as follows:—

Let Fig. 93 be inverted, and let the curve D P C be any meridian of the dome. If a horizontal plane P P′ is passed through the dome, and *all* the weight from the crown to that section is denoted by W, the entire force on the circumferential section will be $W \div \sin\theta$, if θ is the angle which the tangent at P makes with the horizontal. This force, divided by the circumference cut out by the horizontal plane, will be the compression per linear unit of the circumference, or, if divided by the number of ribs in a skeleton dome, will be the thrust in one rib. If the horizontal plane is again passed through a second point a little nearer C, W will increase, the force on the circumferential section will change, since θ changes, and, as the circumference increases, the compression per unit of circumference may be greater, constant, or less, depending on the relative changes in the three factors.

The horizontal component of the entire force in the direction of the tangent at P will be $H = W \cot\theta$. At the section nearer C, H may be still the same numerically. If so, the horizontal band between those two points is in equilibrium, having no tendency to move out or in, and hence having no hoop tension or compression. If, however, H changes, and the change ΔH is a decrease, a force acting inwards must be supplied by that band or hoop, which is then in tension. If ΔH is divided by the circumference, the quotient will be the normal stress per unit of circumference supplied by the ring, and the product of this normal stress and the horizontal radius of the ring will be the tension in the latter, or $\Delta H \div 2\pi$. The stress per linear unit of a meridian section will be found by dividing this expression by the distance between the two horizontal planes measured on the meridian arc. If, on the contrary, ΔH is an increase, the force to be supplied by the band or hoop acts outwards, and compels the hoop to resist compression.

There is usually a decreasing compression in successive hoops from D to a certain section, which for a spherical dome of uniform weight is where the tangent has an inclination of 52° to the horizon, and then increasing tension to C. If the dome does not rise vertically at C, and has vertical reactions, a strong hoop in tension must be supplied at C. At D a circular opening or eye is often made for the admission of light, and this opening may be surmounted by a lantern. A strong ring at the eye is needed to resist compression. The weight of the lantern is easily included with W.

A ribbed dome may be readily treated in this way, and one lune alone considered. The several horizontal planes will then coincide with the purlin rings, and the rib thrusts will be taken as parallel to the chords of the successive segments of the ribs. In this case, as well as in the preceding one, diagrams will be more convenient than calculations, and sufficiently accurate.

If the wind pressure on one side of the dome is likely to be severe enough to be considered, the trapezoidal panels between the ribs and the purlin rings must contain diagonal bracing. As all the horizontal component of the wind pressure above any horizontal plane must be carried past that plane, as a shear, to the supporting wall, the rule may be followed that the maximum shear on a thin circular section is twice the mean shear. Therefore, put diagonal ties in every panel at the level P P′, large enough to carry a force whose horizontal component is equal to the quotient of the horizontal component of all the wind pressure from D to P divided by one-half the number of ribs in the dome. As the wind may blow from any quarter, both diagonals will be necessary.

243. **Resistance of Thin Ring to a Single Load.**—The resistance of sewer pipes and similar hollow cylinders to a single load may be found by the following analysis, the results being applicable to working loads on ductile materials, like steel, and being reasonably correct for breaking loads on such brittle materials as cast-iron and vitrified clay pipes. As the pipes are comparatively thin, as they are often not very homogeneous, and as they vary somewhat from the true circular form, it has not been thought necessary to use the exact

value for the moment of inertia of a hollow cylinder nor to take account of the socket. If the section under trial is moderately long, the socket will have little or no influence on the breaking load.

If a circular pipe were supported at two points of its length and loaded at the middle of the span, the usual formula for the resisting moment, $fI \div y_1$, which would be equated with $\frac{1}{4}Wl$, might be written, if the cylinder is considered to be thin, of a *mean radius* r and thickness t, since $y_1 = r$, and $I = \pi r^3 t$ by § 99, VI.,

$$M = f\pi r^2 t, \text{ or } f = \frac{Wl}{4\pi r^2 t} = \frac{Wl}{4S_0 t},$$

where S_0 = area of circle of radius r.

If, on the other hand, the cylinder rests on the bottom element and is loaded along the top element, it will fail, if brittle, by breaking into four pieces, the lines of fracture running approximately through the top, the bottom, and the extremities of the horizontal diameter. If one will press with his hand on the top of a moderately flexible hoop which rests upright on the ground, he will appreciate the action of a ring or cylinder under two directly opposed equal forces.

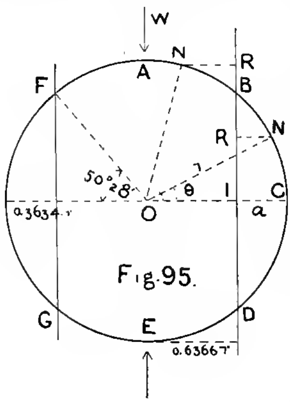


Fig. 95.

To determine the points of zero bending moment or contraflexure, B, D, F and G, Fig. 95, or to locate what then represents the equilibrium polygon, or action line of the forces, it is sufficient to note that the sum of the successive *changes of inclination*, for one quadrant, from A to C, caused by the bending moments at successive points, must equal zero, as the tangents to the curve at these two points must be unchanged in direction. As each change of inclination is directly proportional to the bending moment at the point, the summation of the horizontal ordinates between A and C must be zero. As applied in Graphics, Part III., Arches, let a = C I, the horizontal distance from the ring to the equilibrium line at the

extremity C of the horizontal diameter. Let θ = the angle included between O C and the radius to any point N. Then will N R, the arm of the constant force $\frac{1}{2}$W in B D, be $r(1 - \cos\theta) - a$, which will be of opposite signs for points on either side of B. If this expression is multiplied by $rd\theta$, the length of an infinitesimal arc, and integrated from C to A, its integral must be zero.

$$r\int_0^{\frac{1}{2}\pi}[r(1-\cos\theta)-a]\,d\theta = 0 = \Big[r(\theta-\sin\theta)-a\,\theta\Big]_0^{\frac{1}{2}\pi}$$

$$\tfrac{1}{2}\pi a = (\tfrac{1}{2}\pi - 1)r, \text{ or } a = \frac{0.57}{1.57}\,r = 0.3634r.$$

$$\text{M at side} = -\tfrac{1}{2}W \times 0.3634r = -0.182Wr.$$

$$\text{M at crown} = \tfrac{1}{2}W(1 - 0.3634)r = 0.318Wr.$$

The angle from O C to the points of contraflexure will be $50^\circ\ 28'$, since its cosine is 0.6366. The ratio of the two max. bending moments is 7 to 4, or $1\frac{3}{4}$ to 1, the greatest bending moment being at A and E.

244. Resulting Stresses.—Since the resisting moment of a rectangular section of width l and depth t is $\frac{1}{6}flt^2$, the max. unit tension or compression at the crown will be

$$f_t = \frac{0.318Wr\ .\ 6}{lt^2} = 1.91\,\frac{Wr}{lt^2}.$$

The max. unit compression at the side, on the inside, will be

$$f'_c = \frac{\frac{1}{2}W}{lt} + \frac{0.182Wr\ .\ 6}{lt^2} = \frac{W}{lt}\left(\tfrac{1}{2} + 1.09\,\frac{r}{t}\right).$$

The max. unit. tension at the side, on the outside, will be, if the following expression is positive,

$$f'_t = \frac{W}{lt}\left(1.09\,\frac{r}{t} - \tfrac{1}{2}\right).$$

With ordinary ratios of thickness to radius, the unit stress is much greater at the top and bottom than at the sides, and cracking of the clay or cast-iron pipe may first be expected at the former points; but failure will probably immediately follow at the sides. For a ratio of thickness to radius = $\frac{1}{6}$,

the three values become 68.76, 42.24, and 36.24 $W \div lr$, respectively.

If these values are compared with the stress from the resisting moment for cross-breaking or beam action, it will be seen that the stress at A last deduced exceeds the cross-breaking stress. To make the two stresses equal it is necessary that

$$\frac{Wl}{4\pi r^2 t} = 1.91 \frac{Wr}{lt^2}, \text{ or } l^2 = \frac{24 r^3}{t}.$$

As l in a cross-breaking test of sewer pipe will not exceed twenty inches, r must be about $2\frac{1}{2}\ t$ to satisfy this equation. The fact is thus made clear that sewer pipes, resting on two supports and loaded at mid-span, break in four longitudinal pieces as previously described, not as beams, but as cylinders under two directly opposed forces, and that tests for cross-breaking are not such in fact.

245. Tests of Clay Pipes.—The following results of tests will show the breaking strength as deduced by the formula. All the pipes are from the same maker.

Interior Diameters.		Thickness.	Weight.	Breaking Weight, lbs.	f.
$18\frac{1}{8}$	$18\frac{1}{8}$	$1\frac{5}{16}$ in.	182 lbs.	4,100	1,975 lbs.
$18\frac{1}{4}$	$18\frac{1}{16}$	$1\frac{5}{16}$	$178\frac{1}{2}$	3,900	1,880
$20\frac{7}{16}$	$20\frac{3}{16}$	$1\frac{1}{2}$	$224\frac{1}{2}$	4,750	1,960
24	$23\frac{7}{8}$	$1\frac{5}{8}$	303	4,775	1,965
$24\frac{1}{4}$	$23\frac{7}{8}$	$1\frac{5}{8}$	304	5,200	2,154

The mean diameters and a length of 24 in. were used in calculating f. Pieces of the 20 in. and 24 in. pipes, subsequently broken on two supports with a central load, gave $f = 1,590$ and 1,840 respectively. A piece from the 20 in. pipe, 5 in. long, section $3 \times 1\frac{1}{2}$ in., crushed at 16,000 lbs. per sq. in.

246. Ring under Any Forces.—If four equal forces were applied to the ring of Fig. 95 at points 90° apart, the stresses at A and C on the outside would be added algebraically, and similarly on the inside circumference. As the moments at A and C are of opposite kinds, the new moment would be but three-fourths of that now at C, and about three-sevenths of that now at A. A number of pairs of equal or unequal forces, of the same or opposite kinds, can be readily treated.

From the action of lateral pressure in diminishing the bending moments due to an incumbent load, the ability of a brittle sewer pipe to resist a heavy weight of earth, if good lateral support is

supplied, is made clear. If the pipe is laid in a clay trench, and sand or gravel is not packed around it, the larger sizes are liable to fracture as above described. As, however, the broken pipes cannot spread to any great extent laterally, they may still be serviceable.

The unit earth pressure on different planes, for a given ratio of p_1 to p_2, may be found by § 190. By a similar treatment to that of § 135, the equilibrium curve for a thin ring under earth pressure can then be drawn, and the stress f caused by a given depth of earth can be found. The equilibrium curve, for a pipe buried a considerable distance below the surface of the ground, is elliptical.

This investigation can be extended to rings of other given forms, such as the links of chains, either open or studded. In chain links the form is rather indefinite, the change of form under tension decreases f, and the pull is applied to a considerable portion of the curve, and not at a point. The polygon for a studded link will be a lozenge or diamond.

The ultimate strength of chain is stated to be two-thirds of that of the original bar, and the safe stress for iron is about 12,000 lbs. per square inch of both sides for studded links, and two-thirds as much, or 8,000 lbs. per square inch, for open links. As links differ much in form, only roughly approximate statements can be made.

Examples.—1. A butt-jointed flue in a boiler is 12 in. diam. and 14 ft. long. How thick should it be, if 80 lbs. max. steam pressure is one-sixth of the collapsing pressure? $\frac{5}{16}$ in. +.

2. What is the net thickness required for a boiler shell 60 in. in diameter to carry 120 lbs. steam pressure? What the gross thickness allowing for riveting, and the size and pitch of rivets? $\frac{3}{8}$ in.; $\frac{9}{16}$ in.; $\frac{3}{4}$ in.; 3 rows, 3 in. pitch.

3. What weight applied at top of circumference and resisted at bottom ought a cast iron pipe, 12 in. diam., $\frac{1}{2}$ in. thick, and 6 ft. long, to safely carry, if $f = 12{,}000$ lbs.?

4. Cast-iron, hydraulic cylinder, 3 in. bore, to raise 15 tons. How thick should it be, if unit tension is 7,000 lbs.?

CHAPTER XIV.

PLATE GIRDERS.

247. I Beam.—A rolled beam of I section may be considered composed approximately of three rectangles,—two flanges, each of area A_1, and a web of area A_2. The depth between centres of stress of the flange sections may be denoted by h', which is also very nearly the depth of the web. Then the resisting moment of the two flanges will be $fA_1 h'$, and that of the web, since M for a rectangle is $\frac{1}{6}fbh^2$, is $f \cdot \frac{1}{6}A_2h'$. The value for the entire section will be

$$M = f(A_1 + \tfrac{1}{6}A_2)h'.$$

Hence comes the rule that one-sixth of the web may be added to one flange area in computing the resisting moment of an I beam. The extreme depth of the beam ought not, however, to be used for h'. The approximate distance between centres of gravity of the flanges will answer, since it is a little short of the true value for the flanges and a little longer than is correct for the web.

248. Plate Girder.—A portion of a plate girder and a section of the same is shown in Fig. 96. Such a structure acts as a beam and is designed to resist the maximum bending moments and shears to which it may be liable. It may be loaded on top, or through transverse beams connected to its web. It is used when the ordinary sizes of I beams are not strong enough to resist the maximum bending moment. As the flanges may be varied in section by the use of plates where needed, as shown at the right, there may be more economy of material in using a built beam rather than a rolled one, if the required maximum section is large.

The web A is made of sufficient section to resist the maximum shear, and the rest of the material is thrown into the flanges B, where it will be farthest removed from the neutral axis and hence most efficient in resisting bending moment.

As the thickness of the web plate is usually restricted to one-fourth inch, and in girders of any magnitude to three-eighths inch, as a *minimum*, it appears that the material in the web practically increases with the depth of the girder. As the stress in either flange multiplied by the distance between the centres of gravity of flanges resists the bending moment, the material in the flanges decreases as the depth increases:—hence the most economical depth is that which makes the total material in the web as near as may be equal to that in the two flanges. Depth of beam contributes greatly to stiffness, when a small deflection is particularly desirable, and the depth may in such a case be so great as to make the web the heavier.

249. **Web to Resist Shear Only.**—Some engineers apply the rule of § 247 to a plate girder, that is, add one-sixth

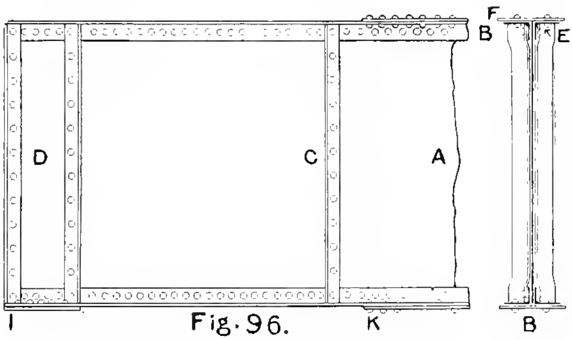


Fig. 96.

of the web section to the flange section for the resisting moment; but the more commonly received practice is to consider the flanges alone as resisting the bending moment at any section and the web as carrying all the shear, uniformly distributed over its cross-section, as shown in § 87. As the riveted connection between the web and the flanges is not so good as solid metal, and as both legs of the angles are included in the cross-section of the flange, with a moment-arm greater than the vertical leg should have, it is advisable not to consider any portion of the web as effective for resisting the bending moments.

250. Compression Flange.—The compression flange must be wide enough not to bend sideways like a strut between points at which it is stayed laterally. As the web checks such lateral flexure in some degree, any column formula to be applied to the flange may be modified to correspond. In some cases the value of a in the denominator is decreased some forty per cent. One authority gives for the allowed unit stress in the compression flange of a railway girder, consistent with safety from lateral deflection, when the flange breadth is b in an unstayed length l, $7,500 \div (1 + \frac{l^2}{417b^2})$ for wrought iron, with a substitution in the numerator of 8,600 for soft steel, and 9,400 for medium steel. For rolled beams, 8,000, 9,200 and 10,000 are specified; and for highway bridges the above values may be increased 25 per cent.

The question is complicated by the fact that the unit stress varies in the flange plate from section to section unless the depth of the girder is made variable to correspond with the change in bending moment. A mean value for the flange stress may be used.

One authority specifies that the compression flanges of beams and girders shall be stayed against transverse crippling when their length is more than thirty times their width. Another prescribes that the unsupported length of compressed flange shall not exceed twelve times its width; and a third specifies the ratio sixteen. Angle irons may be riveted to the edges of the flange to stiffen it, or a channel iron with flanges turned in may be used.

251. Flange Angles.—If the maximum bending moments are computed or obtained by a diagram for a number of points in the span of the girder, they can be divided by the allowable unit stress and the effective depth. The respective quotients will be the necessary net sections of the tension flange at those points The flange angles E must be large enough to support well the compression plates F, if such plates are required, and to be able to transmit the increments of stress from the web to such flange plates. Hence the size of the flange angles should be a considerable portion of the largest net section

found above. For railway girders and floor-beams, it is sometimes required that these angles should have a section at least equal to one-half of the whole flange section required, or be made of the largest angles.

252. **Length of Plates.**—Inspection of the necessary sections will now show how far from the two ends of the girder, as at I K, the flange angles, with rivet holes deducted, will suffice for the required flange section. From K to the corresponding distance from the other abutment the first plate must extend, seen on the right in the figure. A reasonable thickness being used for that plate, with a deduction for rivet holes in the tension flange, it can again be seen where a second plate will be needed, if at all. This determination can be neatly made on a diagram of maximum moments. Extend the plate either way a small additional distance, to relieve the angles and assure the distribution of stress to the plate. The thicker plate, if there is any difference in thickness, should be placed next to the angles.

The compression flange is usually made of the same gross section as the tension flange. The deduction for rivet holes in the latter, which is not necessary in the former, compensates for the slightly lower unit stress allowed for compression. If the girder is so long that the plates or angles must be spliced, additional cross-section must be supplied by covers at the splices, with lengths permitting sufficient rivets to transmit the force. Even compression joints, though carefully dressed and butted together, are spliced, in good practice. The net area of the cover plate and splice angles should be equal to that of the largest piece spliced. Only one piece should be cut at any one section, and enough lap should be given for the use of sufficient rivets to carry the stress the piece would have carried if uncut.

253. **Web: Rivets.**—The web carries the maximum shear at any point, and this shear is uniformly distributed on the vertical section, by § 87. The minimum thickness of web before referred to, $\frac{1}{4}$ or $\frac{3}{8}$ inch, will, for all but long, heavily loaded girders, be sufficient to carry this shear. The unit shear, if specified, runs from 4,000 to 8,000 lbs. per square inch, depending upon the rapidity of imposition of load. As

the equal shears on a vertical and horizontal plane in the web at any point are equivalent to a pull and a thrust of the same unit value at 45° to the horizon, § 189, and Fig. 71, and as these inclined stresses must cause horizontal increments of stress in either chord, one may conceive the web to be divided into square panels and supply enough rivets between the web and the angles, for that panel distance and of uniform pitch, in double shear or in bearing, to transmit a horizontal force equal to the *maximum* shear in the middle of that panel; or the maximum shear at any point of the span may be divided by the depth of the web and the quotient may be considered as the force per unit of length of the flange to be transmitted, from which force the pitch of the rivets may be found. As the rivets through angles and flange plate are in single shear and are in two rows spaced intermediate between those in the vertical legs of the angles, the same pitch will be correct, unless the flange plates are deficient in bearing area. The maximum shear at any point in the span will occur when the longer segment is loaded, if possible.

Make the pitch of rivets in inches and eighths, not decimals; do not vary the pitch frequently, and do not exceed a six inch pitch, so that the parts may be kept in contact. If flange plates are wide, and two or more are superimposed, another row of rivets on each side, with long pitch, may be required, to ensure contact at edges. Care must be taken that a local heavy load at any point on the flange does not bring more shear or bearing stress on rivets in the vertical legs of the flange angles than allowed in combination with the existing stress from the web at that place and time.

Webs are occasionally doubled, making box girders, suitable for extremely heavy loads. The interior, if not then accessible for painting, should be thoroughly coated before assembling.

If the web must be spliced, use strips for that purpose, having the proper thickness for rivet bearing and enough rivets to carry all the shear at that section; do not make a splice with a T iron.

254. **Stiffeners.**—At points where a heavy load is concentrated on the girder, stiffeners C, consisting of an angle

iron on each side, should be riveted, to prevent crushing of the web under the local load and to distribute such load to both flanges. They should for a similar reason be used at both points of support D. These stiffeners may be computed, when necessary, as a + shaped column of four equal arms, each of the width of an angle leg.

Since the thrust at 45° to the horizontal tends to buckle the web, and the equal tension at right angles to the thrust opposes the buckling, it is conceivable that a deep, thin web, while it has more ability to carry such thrust as a column or strut than it would have if the tension were not restraining it, may still buckle under the compressive stress; and it is a question whether stiffeners may not be needed to counteract such tendency. They might be placed in the line of thrust, sloping up at 45° from either abutment, but such an arrangement is never used. They are placed vertically, as at C, and spaced by a more or less arbitrary rule.

A common formula is:—The web of the girder must be stiffened if the shear per square inch exceeds

$$12{,}000 \div \left(1 + \frac{d^2}{3{,}000t^2}\right),$$

where d = clear distance between flange angles, or *between stiffeners* if needed, and t = thickness of web. Another rule calls for stiffeners at distances apart not greater than the depth of the girder, when the thickness of the web is less than one-sixtieth of the unsupported distance between flange angles. The above formula may be written in terms of the slanting distance, the real strut length, but is more convenient as it stands. Experience appears to show that stiffeners are not needed at such frequent intervals as the formula would demand. An insufficient allowance for the action of tension in the web in keeping the compression from buckling it, is probably the cause of the disagreement.

Stiffeners may be offset at the ends, as shown in the figure at E, or filling pieces may be used under the angle stiffeners to avoid the offset; in all cases they should be tightly fitted between the two flanges.

Sometimes the depth of the girder is varied to approximate to the elevation of a beam of uniform strength.

For unit stresses; see § § 171–7.

Example.—A plate girder of 30 ft. span, load 3,000 lbs. per ft., $f = 15{,}000$ lbs. per sq. in. $W = 90{,}000$ lbs., and M max. $= \frac{1}{8}Wl = 4{,}050{,}000$ in. lbs. Assume extreme depth as 42 in., effective depth, 39 in. Net flange section at middle $= 4{,}050{,}000 \div (39 \quad 15{,}000) = 7$ sq. in. A $\frac{3}{8}$ in. web, 42 in. deep, will have $15\frac{3}{4}$ sq. in. area. Two flanges, each 7 sq. in. net + allowance for rivet holes, will fairly equal the web. Use $\frac{3}{4}$ in. rivets.

Let the flange angles be $2 - 4 \times 3 \times \frac{3}{8}$ in. $= 4.96$ sq. in. Deduct 2 holes, $\frac{3}{8} \times \frac{7}{8} = 0.66$. Net plate $= 7 - 4.3 = 2.7$ sq. in. A plate $9 \times \frac{3}{8} = 3\frac{3}{8}$ sq. in.; deduct two holes $= 0.66$, leaving 2.71 sq. in. Two angles and plate, gross section $= 4.96 + 3.37 = 8.33$ sq. in. Resisting moment of net section of angles $= 4.3 \times 15{,}000 \times 39 = 2{,}515{,}500$ in. lbs. Such a bending moment will be found at a distance x from either end, given by $P_1 x - \frac{1}{2} \cdot 3{,}000\, x^2 = 2{,}515{,}500$. $x = 5.9$ ft. $\therefore$ Cut off the plate 5 ft. from each end.

Shearing value of one $\frac{3}{4}$ in. rivet at 10,000 lbs. per sq. in. $= 4{,}400$ lbs. Bearing value in $\frac{3}{8}$ in. plate, at 20,000 lbs. $= 5{,}600$ lbs. Max. shear in web $= 45{,}000$ lbs. Pitch for flange angles, since bearing resistance is less than double shear, $= 5{,}600 \cdot 39 \div 45{,}000 = 4.85 = 4\frac{3}{4}$ in. Make 3 in. pitch for 2 ft., then $4\frac{3}{4}$ in. for 6 ft., then 6 in. pitch to middle. Rivets in end web stiffeners, $45{,}000 \div 5{,}600 = 9$. Max. shear in web $= 45{,}000 \div (42 \cdot \frac{3}{8})$ $= 2{,}780$; $12{,}000 \div \left(1 + \dfrac{d^2}{3{,}000t^2}\right) = 2{,}950$, since $d = 42 - 6$. No other stiffeners needed. By the other rule $36 \div \frac{3}{8} = 96$, and stiffeners are needed.

CHAPTER XV

EARTH PRESSURE: RETAINING WALL: SPRINGS: PLATES.

255. **Pressure of Earth.**—The stability of a mass of earth and the resistance that must be offered by a retaining wall to the thrust of a bank can be determined by the principles of Chap. XI., if it is assumed that the particles of earth are held in place by friction alone. The adhesion arising from the presence of a little moisture is neglected, as always uncertain in amount and sometimes possibly absent. Such adhesion would diminish the pressure against the wall. If the earth is saturated with water, so as to be reduced to mud, it will press normally against the wall as does a fluid, and with a pressure which is to that of water as the weight of a cubic foot of mud is to one of water. If friction alone operates to keep the particles at rest, the greatest possible obliquity of pressure from the normal, consistent with equilibrium, on any plane in the mass of earth, cannot exceed what is known as the *angle of repose;* for, if it did, sliding would take place along that plane.

Let a plane be passed through P, Fig. 97, parallel to the surface of the ground D K. The pressure on every square foot of this plane is vertical, and due to the earth above it, of depth K P. But the prism of earth resting on a square foot of this plane has a smaller horizontal section than one square foot, and the ratio of the unit vertical pressure, on the plane through P, to the weight of a vertical column of earth one square foot in cross-section will be that of the normal P E, drawn from P to D K, to P K. Hence, revolve P E to P G, and G P will represent, in feet of earth, the pressure per square foot of the plane through P parallel to the surface of the ground.

If the principal stresses p_1 and p_2 were known, P M could now be laid off on the normal $= \frac{1}{2}(p_1 + p_2)$, and M G would

be $\frac{1}{2}(p_1 - p_2)$ to close on G P or p, § 190. But, as stated in a preceding paragraph, the greatest obliquity of stress from the normal to any plane cannot exceed the angle of repose of the earth. Hence, if P I is drawn, making that angle with P N, the distance M G must, if applied at M O, make M O P a right angle. Therefore, find by trial a centre M on N P from which a semicircle N G O can be drawn through G and tangent to the line P I. The point M can readily be located very closely. P M will then be $\frac{1}{2}(p_1 + p_2)$ and M G, $\frac{1}{2}(p_1 - p_2)$. By § 193, the direction of p_1 will be parallel to the line M L, drawn bisecting the angle N M G.

It may be noted that the two principal stresses act on the right-angled faces of a small triangular prism at P, the other face being parallel to the surface of the ground; that p_2 is the *least* possible pressure which is consistent with equilibrium, and that it is the one exerted by earth at rest under the action of its own weight only. Blows applied to the surface of the ground, the vibration set up by railway trains, and similar causes will probably increase the pressure and should be allowed for as is a live load on a bridge.

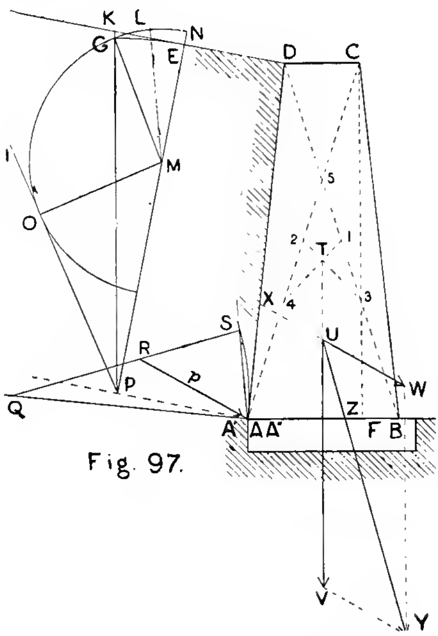


Fig. 97.

256. Pressure Against a Wall.—To find the centre of pressure and maximum unit pressure with its direction on any bed-joint of a retaining wall, weighing w' per cubic foot, pressed at the back by earth of a weight w per cubic foot and of a given angle of repose, proceed as follows:

The bed-joint is A B, Fig. 97, carrying the weight of masonry A B C D, whose centre of gravity is at T. To find

T, draw diagonals A C and B D; bisect each at 1 and 2 respectively; lay off A-4 = C-5 and B-3 = D-5; connect 1 with 4 and 2 with 3; these connecting lines will intersect at the centre of gravity, T. At a point P, the same distance K P below the surface of the ground that A is, make the construction just described in § 255, and bisect the angle N M G by L M.

Find the unit pressure and its direction at A on the plane A D, by § 190, as follows:—Draw A Q perpendicular to A D; lay off A Q = P M = $\frac{1}{2}(p_1 + p_2)$; draw A S parallel to L M, that being the direction of p_1; from Q as centre, with radius A Q, draw an arc cutting A S at S; draw Q S and lay off on it from Q, M G = Q R = $\frac{1}{2}(p_1 - p_2)$; connect R with A, and R A will be the direction of the pressure at A on the back of the wall, and its magnitude per square foot in terms of cubic feet of earth, so that, if R A is measured by the scale of the drawing and multiplied by the weight of a cubic foot of earth, the pressure on the back of the wall per square foot at A will be given.

As the pressure at the back increases regularly with the distance below the surface of the ground, the centre of pressure will be at X, one-third of the slant height from A, and the total earth pressure against one foot in length of the wall will be $\frac{1}{2}$(A D$\times$A R).

257. Resultant Pressure on a Joint.—Draw X U W through X, parallel to R A, and let fall T V vertically through T; make U V = (A B + C D) $\frac{w'}{w}$, and U W = A R $\frac{\text{A D}}{\text{C F}}$.* Complete the parallelogram U V Y W. U Y will be the direction of the resultant pressure on the bed-joint A B, and the point *Z* where it cuts the joint will be the centre of resistance or pressure. The total pressure on the joint will be found by multiplying U Y by one-half the height of the wall above A B and by the weight of a cubic foot of *earth*.

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* The ratio $\frac{\text{A D}}{\text{C F}}$ may be called unity without serious error, unless the wall has a strong batter at the back. By the use of the above factors, U V represents the weight of the wall and U W the total earth pressure at back of same.

258. Maximum Stress on Joint.—If it is thought that this centre of pressure is too near the front for safety, or too near the middle of the joint for economy of masonry, change the section by drawing D A′ or D A″ and try again. A second trial will usually suffice. If the distance B Z is more than one-third of B A the maximum unit pressure per square foot, at B, is, by § 138,

$$p = \frac{2 \,.\, \text{total pressure}}{\text{A B}}\left(2 - \frac{3\text{BZ}}{\text{A B}}\right).$$

If the distance B Z is less than one-third of B A, the maximum unit pressure at B, supposing the cement in the joint to offer no resistance to tension, is

$$p = \tfrac{2}{3} \,.\, \frac{\text{total pressure}}{\text{B } Z}.$$

If this pressure is greater than the masonry can safely resist, make A B wider and try again. The wall will be satisfactory to many engineers if B Z is somewhat greater than $\frac{1}{3}$ A B. A margin is thus left for an increase of pressure beyond the *least* pressure here used. The obliquity of U Y to the perpendicular to A B will determine the tendency of the wall to slide forward. If such sliding seems likely to occur, the bed-joint A B may be inclined backwards. The above constructions are simplified when the surface of the ground is horizontal, and also when it slopes at its angle of repose .

259.—Remarks: Special Case.—A little study of the figure will show that a slope at the back of considerable amount has the advantage of increasing the obliquity of the pressure against the wall, and hence of throwing Z nearer the middle of the joint.

A rough rule for a retaining wall, when the ground surface is level, is to make the *average* thickness one-third of the height. For a wall 15 ft. high, and $2\frac{1}{2}$ ft. wide at coping, this rule would make the base $7\frac{1}{2}$ ft. thick. If the earth to be supported by a wall rests on an inclined stratum which may be penetrated by water to such an extent as to make the inclined plane *slippery*, the component of the weight of all the earth above that plane in a direction tangent to it may be

brought against the wall, and its point of application will be in a line drawn through the centre of gravity of that mass, parallel to the plane of sliding. Such pressure may be too great for any reasonable wall to resist, when it is forced to hold up an entire hillside. Expedients should be resorted to in such a case to thoroughly drain the troublesome stratum, or to build a *bank* of stone and coarse gravel at the toe of the slope behind the wall.

SPRINGS.

260. Straight Spring.—For a spring of varying section, see § 111. If a beam of uniform section, fixed at one end, has a couple or moment applied to it, Fig. 98, in place of a single transverse force, it will, as shown in § 103, bend to the arc of a circle. The deflection will be, if l is the length of the beam, $v = \frac{Ml^2}{2EI}$. Since the stress in the extreme fibre,

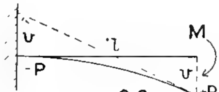


Fig. 98.

$$f = \frac{My_1}{I}, \quad v = \frac{fl^2}{2Ey_1} \text{ and } \rho = \frac{Ey_1}{f}.$$

For very small displacements, the work done by the rotation of the couple M will be, since $M \div l$ is the equivalent force at each end, and dv the small distance through which the force moves at any instant,

$$\text{Work} = 2\int \frac{M}{l}\, dv = \frac{4EI}{l^3}\int_o^v v\,dv = \frac{2EI}{l^3}v^2 = \frac{lI}{2y_1^2}\cdot\frac{f^2}{E}.$$

For a rectangular section bh, these quantities become

$$f = \frac{6M}{bh^2}; \quad v = \frac{fl^2}{Eh}; \quad \rho = \frac{Eh}{2f} \text{ and}$$

$$\text{Work} = bhl \,.\, \frac{f^2}{6E} = \text{Volume} \,.\, \frac{f^2}{6E}.$$

For a circular section the number 6 in the last expression will be replaced by 8.

261. Coiled Spring.—In practice the rectangular or cylindrical bar is bent into a spiral and subjected to a couple $M = Pa$, Fig. 99, which, as a couple can be rotated in its plane without change, acts equally at all sections of the spring. The developed length of the spiral is l.

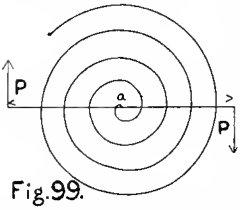


Fig. 99.

262. Helical Spring.—A cylindrical bar whose length is l and diameter d, when fixed at one end and subjected to a twisting moment $T = Pa$ at the other, if the elastic limit is not exceeded, by § 91, is twisted through an angle $\theta = \frac{32Tl}{\pi Cd^4}$. The work expended in the torsion is

$$\int_0^\theta T d\theta = \frac{\pi C d^4}{64 l} \theta^2.$$

From § 91, $\theta = \frac{2q_1 l}{Cd}$; and therefore

$$\text{Work} = \frac{\pi d^2}{4} l . \frac{q_1^2}{4C} = \text{Volume} . \frac{q_1^2}{4C}.$$

If $C = \frac{2}{5} E$ and $q_1 = \frac{4}{5} f$, work = volume . $\frac{2}{5} \frac{f^2}{E}$, while for flexure, as just shown, work = volume . $\frac{f^2}{6E}$, a smaller quantity, so that the torsional moment does more work than the bending moment.

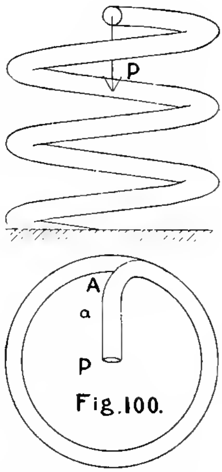


Fig. 100.

If this bar is bent into a helix and the force P is applied at the centre, in the direction of the axis of the cylinder, Fig. 100, to a horizontal arm whose length is a, the arm possessing sufficient stiffness to be not appreciably bent, the moment Pa will twist the bar throughout its length. Then

$$q_1 = \frac{16 \, Pa}{\pi d^3}, \text{ or } P = \frac{\pi d^3}{16 \, a} q_1.$$

The deflection of the spring is $v = a\,\theta$, since, as the force P descends, the spring descends, and the action is the same as if the spring remained in place and the arm revolved through an angle θ. The force P is too small to cause any appreciable compression (or extension) of the material in the direction of its length.

$$v_0 = \frac{32}{\pi} \cdot \frac{Pa^2 l}{C\,d^4} = \frac{2al}{d} \cdot \frac{q_1}{C} = \frac{4\,\pi n a^2}{d} \cdot \frac{q_1}{C},$$

if n = number of turns of the helix, and $l = 2\,\pi a n$.

P may be tension in place of compression.

If the section of the spring is not circular, substitute the proper value of q_1 or the resisting moment from § 92. If the rod is hollow, multiply the exterior volume by $\left(1 - \frac{d'^4}{d^4}\right)$. For a square section and a given deflection, P will be about 65% of the load for an equal circular section. C for steel is from 10,500,000 to 12,000,000.

Example.—A helical spring, of round steel rod, 1 in. diameter, making 8 turns of 3 in. radius, carries 1,000 lbs.

$$q_1 = \frac{16 \,.\, 1{,}000 \,.\, 3 \,.\, 7}{22} = 15{,}273. \quad v = \frac{4 . 22 \,.\, 8 . 9 \,.\, 15{,}273}{7 . 1 \,.\, 12{,}000{,}000} = 1.15 \text{ in.}$$

263. Circular Plates.—The analysis of plates supported or built in and restrained at their edges, and loaded centrally or over the entire surface, is extremely difficult. The following formulas from Grashof's "Theorie der Elasticität und Festigkeit" may be used. The coefficient of lateral contraction is taken as $\frac{1}{4}$, or $m = 4$.

I. Circular plate of radius r and thickness t, supported around its perimeter and loaded with w per square inch.

f_x = unit stress on extreme fibre in the direction of the radius, at a distance x from the centre.

f_y = unit stress perpendicular to the radius, in the plane of the plate, at the same distance x from the centre.

$$f_x = \frac{45}{128}\,\frac{w}{t^2}\left(\tfrac{13}{5}\,r^2 - 3x^2\right); \quad f_y = \frac{45}{128}\,\frac{w}{t^2}\left(\tfrac{13}{5}\,r^2 - x^2\right).$$

$$f_x = f_y \text{ max. (for } x = 0) = \frac{117}{128}\,\frac{wr^2}{t^2}. \quad v_0 = \frac{189}{256}\,\frac{wr^4}{Et^3}.$$

For the same value of t, the max. stress is independent of r, provided the total load $w\pi r^2$ is constant.

II. Same plate, built in or fixed at the perimeter.

$$f_x = \frac{45}{128}\frac{w}{t^2}(r^2 - 3x^2); \quad f_y = \frac{45}{128}\frac{w}{t^2}(r^2 - x^2).$$

At the centre, $f_x = f_y$. At the circumference f_y is zero, and f_x is max.

$$f_x \text{ max.} = \frac{45}{64} \cdot \frac{wr^2}{t^2} . \quad v_0 = \frac{45}{256}\frac{wr^4}{Et^3} .$$

III. Circular plate supported at the perimeter and carrying a single weight W at the centre. Loaded portion has a radius r_0.

$$f_x = \frac{45}{32\pi}\frac{W}{t^2}(\log\frac{r}{x} - \tfrac{1}{5}); \quad f_y = \frac{45}{32\pi}\frac{W}{t^2}(\log\frac{r}{x} + \tfrac{4}{5}).$$

These expressions become maxima for $x = r_0$, and the second is the greater.

$$v_0 = \frac{117}{64\pi}\frac{Wr^2}{Et^3} .$$

For values of $r \div r_0 =$	10,	20,	30,	40,	50,	60,	
f max. $=$	1.4	1.7	1.9	2.0	2.1	2.2	$W \div t^2$.

If $r_0 = 0$, the stress becomes infinite, as is to be expected, since W will then be concentrated at a point, and the unit load becomes infinitely great. It is not well to make r_0 very small.

IV. Same plate, built in or fixed at the perimeter.

$$f_x = \frac{45}{32\pi}\frac{W}{t^2}(\log\frac{r}{x} - 1); \quad f_y = \frac{45}{32\pi}\frac{W}{t^2}\log\frac{r}{x}$$

The maximum value of f is f_y, for $x = r_0$.

$$v_0 = \frac{45}{64\pi}\frac{Wr^2}{Et^3} .$$

For values of $r \div r_0 =$	10,	20,	30,	40,	50,	60,	
f max. $=$	1.0	1.3	1.5	1.6	1.7	1.8	$W \div t^2$.

264. Rectangular Plates.—The problem of the resistance of rectangular plates is more complex than that of circular plates. Grashof gives the following results:

V. Rectangular plate of length a, breadth b and thickness t, $a > b$, built in or *fixed* at edges and carrying a uniform load of w per square inch.

$$f_a = \frac{b^4 \,.\, wa^2}{2\,(a^4 + b^4)\,t^2}\,;\quad f_b = \frac{a^4 \,.\, wb^2}{2\,(a^4 + b^4)\,t^2}\,.$$

The most severe stress occurs at the centre in the direction b, that is, on a section parallel to a.

$$\text{If } a = b,\quad f = \frac{wa^2}{4t^2}\,.$$

The deflection at the centre is $v_0 = \dfrac{a^4\,b^4}{a^4 + b^4} \cdot \dfrac{w}{32\,\mathrm{E}t^3}$, and for a square plate, $\dfrac{wa^4}{64\,\mathrm{E}t^3}$.

VI. Plate carrying a uniform load of w per square inch and supported at rows of points making squares of side a. Fire-box sheet with staybolts.

$$f = \frac{15}{64}\,\frac{wa^2}{t^2}\,;\quad v_0 = \frac{15}{512}\,\frac{wa^4}{\mathrm{E}t^3}.$$

Navier gives formulas for rectangular plates which are supposed to be very thin. Approximate values from those formulas are as follows:

VII. Rectangular plate, as in V., but *supported* around the edges.

$$f = 0.92\,\frac{a^4 b^2}{(a^2 + b^2)^2}\,\frac{w}{t^2}\,;\quad v_0 = 0.19\,\frac{a^4 b^4}{(a^2 + b^2)^2}\,\frac{w}{\mathrm{E}t^3}.$$

VIII. Rectangular plate, supported at edges and carrying a single weight W at centre.

$$f = 2.28\,\frac{a^3 b}{(a^2 + b^2)^2}\,\frac{\mathrm{W}}{t^2}\,;\quad v_0 = 0.46\,\frac{a^3 b^3}{(a^2 + b^2)^2}\,\frac{\mathrm{W}}{\mathrm{E}t^3}.$$

For the same total load, f is independent of the size of the plate, provided the ratio a to b and the thickness are unchanged.

Example.—A steel plate, 36 in. square and $\frac{1}{4}$ in. thick, supported at edges, carries 430 lbs. per sq. ft., or 3 lbs. per sq. in. $f = 0.92 \quad \frac{1}{4} \quad 36 \cdot 36 \cdot 3 \cdot 16 = 14{,}300$ lbs.

$$v = \frac{0.19}{4} \cdot \frac{36^4 \cdot 3 \cdot 4^3}{30{,}000{,}000} = \tfrac{1}{2}\text{ in.}$$

265. **Corrugated Iron.**—The curves to which corrugated sheets are bent can be defined only by their depth and width. Since it is necessary to determine the moment of inertia of the cross-section of the sheet transversely to the corrugations, in order to write an expression for the resisting moment, or to determine the stiffness, any curve may be assumed which will be a good approximation to the probable one, and for which the moment of inertia can be obtained in the above-mentioned terms. The cycloid is such a curve.

Let A B C, Fig. 101, be the curve described by the point B of the circle B D, as it rolls on the line A D C. Any arc B I of the cycloid will be equal to twice the length B H of the chord of the generating circle drawn from B to the point H which B will occupy on the circle, when in rolling it has reached I of the cycloid. For, as the circle turns at any instant about the point of contact with A D C, the point H moves along H K. with radius D H, for a small displacement, describing I N. But H K is equal to twice the amount by which B H′ exceeds B H; therefore any arc B I is equal to twice the corresponding chord B H.

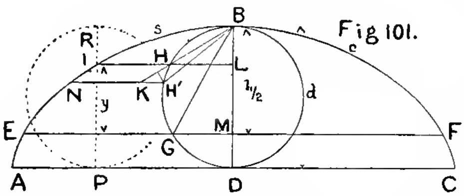


Fig 101.

Let $BI = s = 2BH$; $BF = c = 2BG$.

$BH^2 = BL^2 + LH^2$, and $LH^2 = BL \cdot LD$. Therefore

$$BH^2 = BL^2 + BL \cdot LD = BL(BL + LD) = BL \quad d,$$

where d = diameter of generating circle. Similarly

$$BG^2 = BM \cdot d. \quad \text{But } BE \cdot BF = c^2 = 4BG^2 = 4BM \cdot d.$$

$$EI \cdot IF = (c - s)(c + s) = c^2 - s^2 = 4(BG^2 - BH^2)$$
$$= 4(BM - BL)d.$$

$$\frac{c^2 - s^2}{c^2} = \frac{BM - BL}{BM} = \frac{y}{\frac{1}{2}h}, \text{ or } y = \frac{h}{2}\left(1 - \frac{s^2}{c^2}\right).$$

Thus it appears that any portion of a cycloidal arc above an *arbitrary line* E F, drawn parallel to A C, may be taken, and the ordinate y from that line to any point I has the ratio

to the ordinate at the middle, $\frac{1}{2}h$, of the product of the respective segments I E and I F to B E and B F, into which those ordinates divide the arc E B F. Then, for arc E B, of thickness t,

$$\mathrm{I} = t\int y^2 ds = \frac{th^2}{4}\int_o^c \left(1 - \frac{s^2}{c^2}\right)^2 ds = \frac{th^2}{4}\left[\left(s - \frac{2s^3}{3c^2} + \frac{s^5}{5c^4}\right)\right]_o^c = \frac{2}{15}h^2 ct.$$

As ct = cross-section, $r^2 = \dfrac{2}{15}h^2$.

$$\mathrm{M} = \frac{f\,\mathrm{I}}{y_1} = \frac{4}{15}fhct.$$

The depth of the corrugation is h, the breadth of the sheet in the undulating line is c. The breadth straight across may be used by modifying slightly the value of f.

CHAPTER XVI.

DETAILS IN WOOD AND IRON.

266. **General Principles.**—In designing and executing all kinds of joints and fastenings the following general principles, as given by Rankine, should be used as a guide:—

To cut the joints and arrange the fastenings so as to weaken the pieces of timber they connect as little as possible.

To place each abutting surface in a joint as nearly as possible perpendicular to the pressure which it has to transmit.

To proportion the area of each such surface to the pressure which it has to bear, so that the timber may be safe against injury under the heaviest load which occurs in practice; and to form and fit every pair of such surfaces accurately, in order to distribute the stress uniformly.

To proportion the fastenings, so that they may be of equal strength with the pieces which they connect; and to place them so that they may not shear out of the timber nor crush the fibres.

The same principles are applicable to metallic construction.

267. **Framing of Timber: Splices.**—Sketches XI. to XVI. in Plate II. represent different methods of splicing a timber tie. In each case the smallest cross-section of the timber determines the amount of tension that can be transmitted. The shoulders are in compression, and the longitudinal planes between the shoulders are in shear. In XI., for equal strength, the depth of the two opposite shoulders or indents should be to the remaining depth of the timber as the safe unit tensile stress is to the safe unit compression along the grain. The shearing length, on either timber or clamp, should be to the depth of shoulder as the safe unit compression is to the safe unit shear. In actual practice, unless considerable dependence is placed upon the resistance of the

bolts against shearing through the timber, the splice should be much longer than shown. If the two clamps are of stronger wood than the main timber, they need not together have so much depth as the net depth of the timber. The iron strap in XIV. illustrates the same principle. The bolts are usually small, and serve mainly to balance the couples set up on each clamp by the pressure on the shoulder and the tension in the neck. The modification in XII. permits the introduction of the bolts without reducing the net section of the timber. In XIII., each indent is only half the previous depth, with obvious economy of the main timber, and increase of shearing area of clamp and timber without lengthening the clamps. It is much more difficult to fashion, however, and it is not probable that both shoulders on one half will bear equally.

XV. and XVI. are scarfed joints. The tension sections, the compression shoulders and the longitudinal shearing planes should again be properly proportioned here. In XV., but one-third of the timber is available, if unit tension and compression have the same numerical value, while in XVI. one-half of the stick is useful; but the latter joint is more troublesome to fashion. The bolts serve to resist the couple which tends to open the joint, and, by resisting it, cause a fairly uniform distribution of stress in the critical section. The bolt holes do not weaken the timber. Sometimes the extreme ends of the scarf are undercut to check the tendency to spring out when the bolts are not used. Keys may be driven through places cut for them at the shoulders. The joint can then be readily assembled and forced to place. These sketches show that timber, although possessing good tensile strength, is ill-adapted for ties, on account of the great loss of section in connections and joints.

268. **Struts and Ties.**—The connection of a strut and tie in wood is illustrated in II., III., IV. and VII. The shrinkage of the pieces of II. in seasoning tends to open both portions of the joint by changing the angles; but the bearing of the strut is still central, if only on a small area. The compression of the tie across the grain may be large in such a case, and the introduction of a block, as in IV., will remedy such a difficulty as well as that from shrinkage. The block below is

the wall-plate, for distributing the truss load along the wall. It is subjected to compression across the grain.

If the shearing area to the left in these four cases is not sufficient, the bolt or strap is a wise provision to take up the horizontal component. The bolt, if a little oblique to the strut, as shown, holds at once by tension, to some degree, and not alone by shear. It also relieves the smallest section of the tie from a part of the tension. The square shoulders of III. are good, if the timber is seasoned, as the bearing is then over the whole end of the strut, and the tie is not weakened any more than in II., while the joint is more simply laid out. The strap of VII. gives a satisfactory bearing for the strut, but the fastenings of such a strap are often weaker than the strap itself. The holes in it may well be enlarged hot, without removal of metal and diminution of cross-section.

In VIII., IX. and X. are shown connections of struts which may at some time be called on to resist tension, or which may be relieved of stress and become loose. The tenon in VIII. must be pinned to carry tension; and the pin will resist but little before shearing out of the tenon or splitting off the side of the other timber by tension across the grain. The tenon should be fashioned as indicated, with sufficient area at the left hand edge to carry the perpendicular component of the thrust of the strut as compression across the grain, and sufficient cross-section not to shear off. The size of the strut must be determined, not only by the column strength, but by the area necessary to prevent crushing the piece against which it abuts. This remark applies to IX. and X. also. The ability of IX. to carry tension depends on the resistance of the nut, which is slipped into a hole at the side, to shearing out along the strut, or crushing the fibres on which it bears, the latter method of failure being the more likely, unless the nut is quite near the end of the strut. The strap on X. is very effective, and the arrangement, if inverted, will serve as a suspending piece, although a rod is better. Many of these connections are serviceable in other positions.

To keep a strut from crushing the side of a timber, a connection may be employed, as in the lower part of I. This device may be economical, if a number of such joints are to

be made, and it is superior to a mortise in work exposed to the weather, as there is no place for water to lodge. The post in XVIII. is capped by a similar device for distributing and thus reducing the unit pressure on the other piece. Lateral displacement is provided against in both cases by ribs on the castings.

Strut connections are shown in XIX. and XX., with a tie rod in addition. The broad, flat washer reduces the unit compressive stress on the wood under it: the lip keeps water out of the joint. Shrinkage and a slight deflection of the frame under a load will cause the mitre joint in XIX. to bear at the top only, throwing the resultant stress out of the axis of the respective compression members, §§ 137, 151, and causing the unit compression at top edge of the joint to be very high. The joint in XX. gives a better centre pressure, and is easily made; the upper piece is simply notched for one-half its depth, and the upper and lower edges come on the mitre line of XIX. The connection of XXI., by the insertion of an iron plate or a block of wood, secures a certain continuity or rigidity in the joint, to resist a moderate amount of bending moment. The two pieces might have been halved together. XXVI. is like VIII., without provision for tension, which is usually unnecessary. The roof purlin with its block is also shown in relative position.

269. **Beam Connections.**—In I. and XVIII. are shown supports of beams on posts. The double or split cap of I. is serviceable where several posts are to be connected laterally, as in a trestle bent, and it is desired to do away with mortises. A mortise and tenon of usual proportions are shown in XVII. Bolts should be put transversely through the caps and top of the post. A comparatively wide bearing for the beam, without the use of large timber caps, may be here secured. Lateral bracing, as in XXVIII., will be needed. An indirect and intermediate support for a beam, by two inclined braces, is seen in XXV., and the reverse case is represented in XXVII. The ordinary wall bearing for joists may be seen in the lower left-hand corner. The slanting end is a wise provision to prevent harmful action of the loaded joist on the wall, and it promotes ventilation of the timber.

The usual way of connecting two floor joists or beams, when their upper surfaces are to be at one level, is drawn in VI. The nearer the mortises are to the neutral axis, the less the weakening of the pieces in which they are cut; on the other hand, the farther the two tenons are apart, the more firmly is the tenoned joist held against lateral twist. The shouldered tenon, indicated by the dotted lines at the left, is designed to attain both objects, to weaken the mortised piece as little as possible and to have a considerable depth of tenon, as well as a long tongue projecting entirely through. The work of framing is considerably more than in the former case.

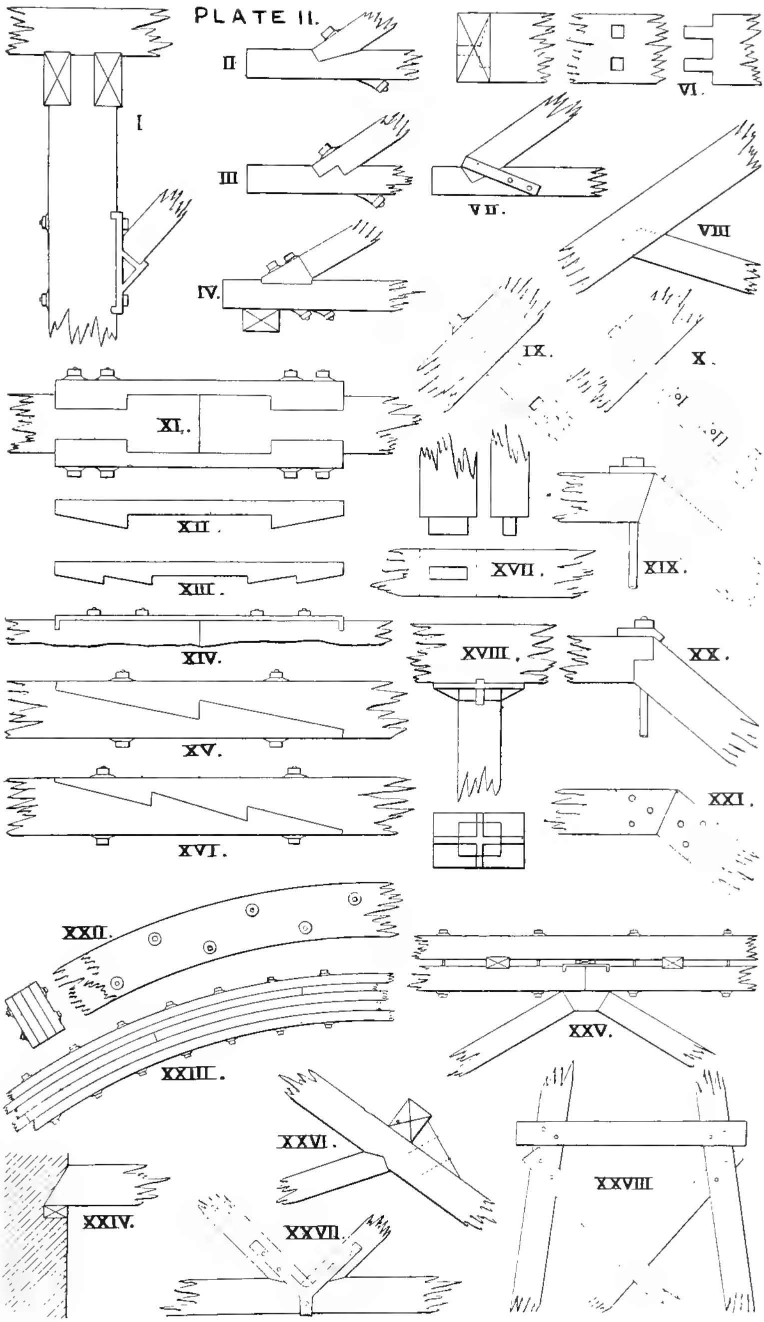
270. Wooden Built Beams.—If seasoned material is at hand, and large timbers are too expensive, a useful beam may be built up by placing planks, from two to four inches thick, edge to edge, and then thoroughly nailing or spiking boards on both sides at an angle of 45° with the length of the beam, and sloping in opposite directions on the two sides. The planks will carry the direct stress due to the bending moments, and the boards will resist tension and compression equivalent to the shear, as shown in § 186. By due regard to jointing and nailing a beam of considerable span may be made at moderate cost. The construction can be doubled if necessary.

Another compound beam is seen in XXV The keys and bolts resist the shear along the neutral axis; the horizontal sticks are butted together on the compression side, and are strapped by the metal clamp indicated to carry tension, if necessary. The small block behind the clamp keeps it in place.

A timber beam has been fashioned like a plate girder, with a close web of diagonal boards spiked at 45°, and flanges of planks connected by other planks occupying the place of the flange angle irons. It's efficiency is uncertain. The old-fashioned plank lattice bridge was very cheap, where lumber was plenty, as the cost of construction was very small.

271. Curved Beams.—Planks placed side by side, as in XXII., cut to the form of a curved beam or arched rib, and bolted together to prevent individual lateral yielding, are quite effective, if the grain of the wood does not cross the curve too obliquely. Hence, when the curvature is considera-

PLATE II.


I
II
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XXIV.
XXV.
XXVI.
XXVII.
XXVIII

ble, it may be advisable to use short lengths, which must break joint in the several parallel pieces. It is well to make a deduction of one piece in computing the strength of the member at any section. The ratio of strength of this combination, when well bolted together, to that of a solid stick may be considered to be as $n - 1$ to n, where n is the number of layers.

If the planks are bent to the curve and laid upon one another, as in XXIII., this combination is not nearly so effective as the former, but it can be more cheaply made. The lack of efficiency arises from the unsatisfactory resistance offered to shear between the layers by the bolts or spikes. The strength to resist bending moment will be intermediate between that of of a solid timber and that of the several planks of which it is composed, with a deduction of one for a probable joint. It may be taken, if the beam is well bolted, as the mean of the two values, or as $(n^2 + n - 1) \div 2n^2$ of the resistance of a solid stick, if $n =$ number of layers, or as $(n + 1) \div 2n$, when one layer is not deducted.

Example.—An 8 in. by 8 in. beam is made of four 2 in. planks on edge, with a joint every 3 ft. Its resisting moment will safely be $\frac{1}{6}fbh^2 \,.\, (n - 1) \div n = \frac{1}{6}f\, 8^3 \,.\, \frac{3}{4} = 64f$.

A similar curved beam is made from eight 1 in. boards, bent to curve and well nailed, one on top of the other. Its resisting moment will be $\frac{1}{6}fbh^2.\ (n^2 + n - 1) \div 2n^2 = \frac{1}{6}f\, 8^3 \,.\, 71 \div 128 = 47\frac{1}{3}f$.

If the curved member has a direct force acting upon it and a moment arising from its curvature, the treatment will follow the same lines; but the joints, if there are any, will be more detrimental in case there is tension at any section. Such curved pieces are sometimes used in open timber trusses for effect, but their efficiency is low on account of the large moment due to the curvature. XXII. is the stiffer.

The joints and connecting parts in all timber construction should be proportioned in detail for such tension, compression and shear as they may have to withstand. Often the three kinds of stress occur in different parts of one joint or connection.

272. **Iron Roof Truss.**—Joints I. to IV., Plate III., represent ways of connecting the several pieces of a comparatively light roof-truss. All the members are made with angles

and at several points both legs of the tension angles are fastened. Joint I. comes between II. and III., and IV. comes perpendicularly opposite it. The number of rivets in each of the ties and centre member of II. depends upon the force in the particular piece and the rivet shearing value and bearing value in the thinnest piece. The number of rivets in the rafter likewise depends upon the force it carries, unless the two rafters are supposed to abut and to transmit so much of the horizontal component as does not come through the inclined ties, a treatment not to be commended. The two angle-irons of the rafter, being in compression, should be connected at intervals by a rivet and filling piece or thimble. The number of rivets through the rafter and connection plate at I. need only be enough to transmit the force from one diagonal to the rafter. Study the necessity for rivets, and do not add all the rivets in abutting pieces to obtain the number in a main member.

Similarly, in IV., the first four or possibly five rivets on the left in the horizontal member balance the rivets in the inclined tie on the right; the six remaining rivets seen and three others unseen, on the left of the splice, balance the same number in the smaller angle. Note how, by an extension of the connecting plate and a short plate below, the main tie is neatly spliced and reduced in section.

The rafter at III. has more rivets than at the upper end because the thrust is somewhat greater. The rivets in the tie at that connection will practically equal those at the other end of the same piece. The black holes at VI. indicate the rivets to be inserted at the time of erection, and these should, in good practice, exceed the number called for in joints riveted in the shop. They must carry the load and resist the moment of the horizontal component due to the wind pressure, which passes down the post IX. as shear. The post is subjected to bending moment as well as compression, and hence has one dimension much greater than the other. Bracing perpendicular to the plane of the truss is needed to resist wind pressure on the end of the structure. Columns and compression members, in structural work of any kind, if joined one to another, must be thoroughly stayed against lateral movement.

Pin-connected roof trusses resemble in their details the joints of the next section.

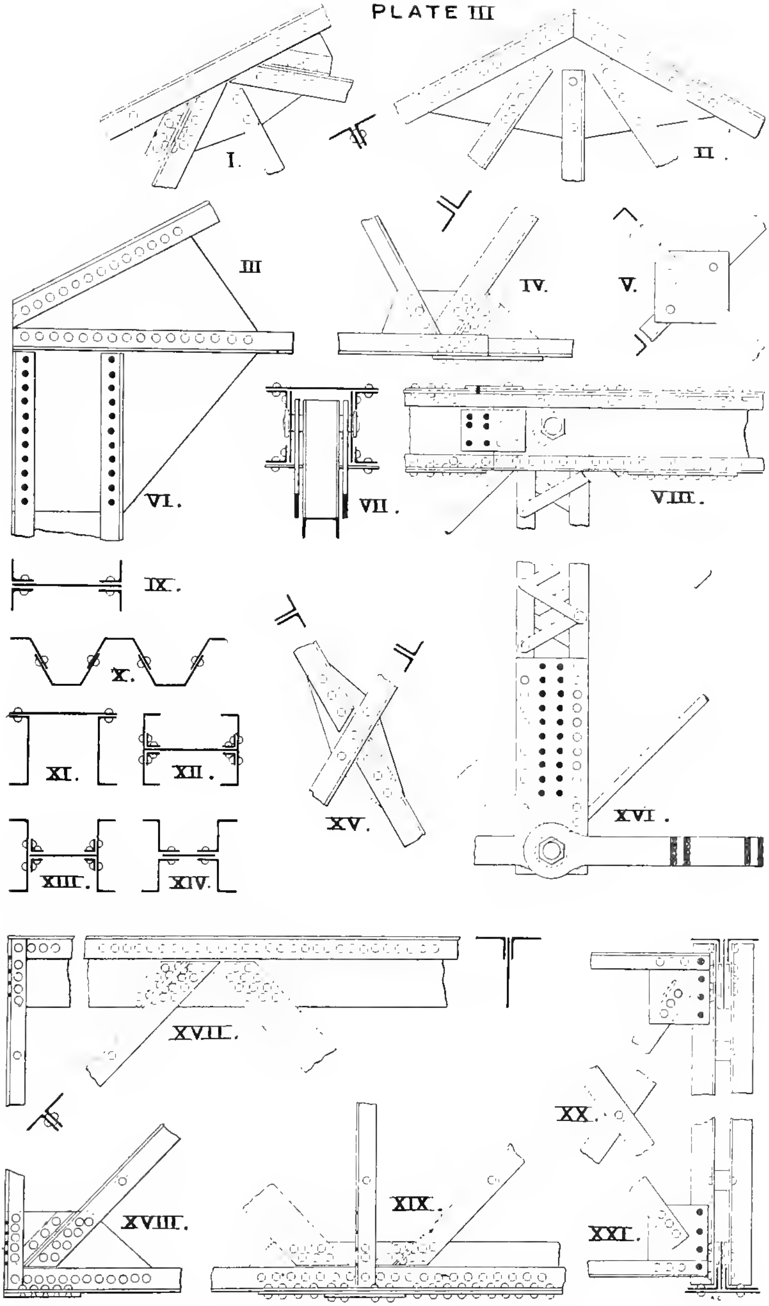
273. **Pin-Connected Bridge.**—Ordinary details in a pin-jointed bridge truss of moderate span are shown in VII., VIII. and XVI. The position of the splice in the top chord is near the pin. The splice plate may be extended to reinforce the pinhole, if required. The ends of the chord pieces are machined plane and parallel, and only enough rivets are then used in the splice to insure the alignement. The pin is placed in the centre of gravity of the chord section, unless slightly changed from that position for the same reason as is given under § 130. The connection plates are seen below, to keep the sides of the chord from spreading; the rest of the panel length is usually laced. Another chord section, employing channels, is drawn at XI.

The post has been discussed in Chapter IX. XII., XIII. and XIV. show other sections for posts. They offer facilities for the central support of floor-beams. Post flanges are sometimes turned out, sometimes in. The floor-beam, of plate girder type, is riveted at XVI. to the post through the holes shown. This attachment stiffens the trusses laterally and is much superior to hangers. Top and bottom lateral bracing, to convey the wind pressure to the abutments, is needed in the planes of the chords, and portal bracing at each end to throw the wind pressure from the top system into the end posts, which convey it to the abutments as shear, with the accompanying bending moments in those posts.

The posts go inside of the top chord, as do the main diagonals or ties, which come next to the posts. The bottom chord bars are on the outside, one of those running towards the middle of the span being usually the farthest out. The bending moment on the pin was discussed in § 224.

274. **Riveted Bridges.**—A riveted Warren girder or latticed truss is shown below. These details are not for consecutive joints. The increase of chord section, when necessary, is indicated at XIX. If the truss is loaded on the top, interior diagonal bracing, drawn at XXI., must be used. When the truss is a lattice, the web members are connected at intersections to stiffen the compression members, as at

PLATE III



I.
II.
III
IV.
V.
VI.
VII.
VIII.
IX.
X.
XI.
XII.
XIII.
XIV.
XV.
XVI.
XVII.
XVIII.
XIX.
XX.
XXI.

XX., or preferably as at V., or at XV , if the web is double. Horizontal lateral bracing must not be overlooked.

X. is one form of section of a solid bridge floor. Rectangular sections are also used.

Examples.— 1. Four timbers, 6 in. $\times$ 12 in., 15 ft. long, are in compression. If placed side by side, with space between for circulation of air, what will be the max. permissible distance between packing pieces, so that the timbers may be equally safe against flexure in either direction?

2. What pull can the bolt of IX., Plate II., safely resist, if the nut is $1\frac{1}{2}$ in. square, is 8 in. from end of stick, and $q = 150$ lbs.? If the compression under the nut ought not to exceed 1,800 lbs., what can the bolt carry without deducting its cross-section?

7,200 lbs. 4,050 lbs.

3. An 8 in. by 8 in. timber of Southern pine is spliced as in XI., Plate II. Design the splice by § 169, 4th line from bottom of table, and find what it will carry, neglecting the bolts.

4. A cast beam, 8 ft. long, supported at ends, has to carry 900 lbs. per ft. If $f_t = 5{,}000$ lbs., $f_c = 13{,}500$ lbs., and the section is I shaped, all parts $\frac{1}{2}$ in. thick, what will be the breadth of top and bottom flanges to resist all the bending moment, if the effective depth at middle is 8 in.? If the flange sections are constant, what will be the elevation of the beam, and the depth at quarter span?

1.6 in., 4 32 in.; 6 in.

5. A vessel is 200 ft. long. It carries 5 tons per ft. uniformly distributed, and a central load of 300 tons. Find M max. when at rest; when supported on a wave crest at bow and stem with each bearing 20 ft. long; and when supported amidships only with bearing 30 ft. long.

6. The end of a beam 6 in. wide is built into a wall 18 in. The bending moment at the wall is 600,000 in. lbs. If the top of the beam bears for 9 in. with a uniformly varying pressure and the bottom the same, what is the max. unit compression on the bearing surface?

1,852 lbs.

7. A plate girder for draw-span, 8 ft. 3 in. in outside depth, has a web of section, $96 \times \frac{3}{8}$ in., weight 120 lbs. per ft.; 4 flange angles, each $6 \times 6 \times 1$ in., 39.2 lbs. per ft., I for one angle = 43.1, distance of centre of gravity from back of angle 1.96 in.; 15 in. channel on top and bottom, 60 lbs. per ft., 1 in. web, flanges turned in, I = 23.0, distance of centre of gravity from back, 0.95 in.; and a $15 \times \frac{1}{2}$ in. flange plate of 25 lbs. on each flange. Sketch the section, find I, and the weight per ft. Add 360 lbs. per ft. If the girder is 95 ft. long and fixed at one end only, what are f and v max.?

I = 247,447; $w = 447$ lbs.; $f = 8{,}740$ lbs. $v = 2.2$ in.

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